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$N = 2$ boundary conditions for non-linear sigma models and Landau-Ginzburg models

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Abstract: We study $N = 2$ nonlinear two dimensional sigma models with boundaries and their massive generalizations (the Landau-Ginzburg models). These models are defined over either Kähler or bihermitean target space manifolds. We determine the most general local $N = 2$ superconformal boundary conditions (D-branes) for these sigma models. In the Kähler case we reproduce the known results in a systematic fashion including interesting results concerning the coisotropic A-type branes. We further analyse the $N = 2$ superconformal boundary conditions for sigma models defined over a bihermitean manifold with torsion. We interpret the boundary conditions in terms of different types of submanifolds of the target space. We point out how the open sigma models correspond to new types of target space geometry. For the massive Landau-Ginzburg models (both Kähler and bihermitean) we discuss an important class of supersymmetric boundary conditions which admits a nice geometrical interpretation.

Keywords: D-branes, Extended Supersymmetry, Differential and Algebraic Geometry.

Contents

1. Introduction

The superconformal boundary conditions that arise from the two-dimensional non-linear sigma model are relevant for the description of D-brane in a semiclassical approximation. Thus the superconformal boundary conditions of sigma model are of manifest interest for string theory.

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In the present paper we undertake a study of local $N = 2$ superconformal boundary conditions for the general (2,2) non-linear sigma model. There are two different types of (2,2) non-liner sigma model depending on the presence or not of a torsion $H = dB$. In the torsion free case the (2,2) sigma model has to be defined over a Kähler manifold [1]. The requirement of conformal invariance (i.e., Ricci flatness) restricts these manifolds to be Calabi-Yau manifolds which play a prominent role in string compactification. Starting with the work of [2], $N = 2$ superconformal boundary conditions for Kähler sigma models have been extensively discussed in the literature. However it is instructive to return to the problem and rederive the known results in a systematic fashion. Indeed we find new properties of the $N = 2$ superconformal boundary conditions for the Kähler sigma model. We believe that our treatment of the Kähler sigma model is exhaustive within our framework and given our general assumptions.

In the torsionful case the $N = (2, 2)$ sigma model has to be defined over a bihermitean manifold with special properties [3]. This type of manifold can be thought of as a special kind of modified Calabi-Yau manifold [4] and may play an important role for string compactifications in the presence of non-zero NS-NS three form field stregth (e.g., the $N = 2$ WZW models). We do not know of any previous study of the $N = 2$ superconformal boundary conditions for the sigma model over a bihermitean manifold and present a detailed analysis of this case.

There are massive generalizations of $N = 2$ non-linear sigma models (the Landau-Ginzburg models) which are (manifestly) not conformally invariant. However they admit a wide class of supersymmetric boundary conditions which are similar to the D-brane conditions for the conformal sigma model. Hence it is appropriate to discuss the Landau-Ginzburg models in the present context (For $N = 1$, boundary conditions are discussed in [6]). By analysing $N = 2$ Landau-Ginzburg models we clarify certain points concerning previous results [5] and also to extend the analysis to the bihermitean Landau-Ginzburg models.

Before going into the detailed discussion of the problem let us briefly examine the notion of a boundary condition which respect some symmetries (e.g., supersymmetry). This is a much more subtle question than usually appreciated.

We identify three different routes for finding boundary conditions: By

- requiring the boundary conditions to be consistient with the symmetry algebra, e.g. for (on-shell/off-shell) supersymmetry

$$F(X, \psi) = 0 \qquad \rightarrow \qquad \delta_{\text{susy}} F(X, \psi) = 0$$

- imposing appropriate boundary conditions on conserved currents in order to have the conserved charges in the presence of boundaries, e.g. for $N = 1$ superconformal symmetry

$$T_{++} - T_{--} = 0 \qquad G_{+} - \eta G_{-} = 0$$

- requiring that the boundary terms of the appropriate variations of the action are zero (e.g., the general field variation and the supersymmetric variation)

$$\delta S = 0\,, \qquad \delta_{\text{susy}} S = 0\,.$$

It is important to realize that there are clear distictions between these three alternatives. The first algebraic requirement is extremely useful since it relies entirely on the representation theory of a given symmetry. For instance, as we will see in the case of supersymmetry there are properties of the boundary conditions which are found using only off-shell supersymmetry. Therefore these properties do not depend on the details of the model. However one cannot hope to get all dynamics out of the algebra and thus in this respect the algebraic requirements are not complete.

On the other hand the conditions involving currents are dynamical requirements which depend only on the bulk properties of the theory. The currents are defined only on-shell and thus the boundary conditions for the currents make sense only on-shell.

From a technical point of view the action is, perhaps, the simplest way to derive boundary conditions. But using an action is difficult in a generic situation since we have to more or less guess which boundary terms to include. An example of this difficulty is the $N = 1$ sigma model with non-zero B-field [7]. Starting from the standard $N = 1$ superfield action one would not find boundary conditions which respect supersymmetry on the boundary. One would then have to add boundary B-field terms by hand. Once one has found boundary conditions using one of the other approaches, however, one may add boundary terms to the action so as to reproduce them. Because of these issues there are contradictory statements about boundary conditions for the $N = 1$ sigma model with B-field in the literature (see, for example, [8]).

Consequently these approaches have their own advantages and disadvantages. Nevertheless these three approaches (or combinations of them) may produce a reliable answer. In this article we adopt the following logic. We always start from the analysis of the algebraic requirements on the general ansatz for boundary conditions on the physical fermions and subsequently study the boundary conditions for currents. As a final step in the analysis we find an action from which the boundary conditions can be derived. To avoid confusion we discriminate between the *bulk action* (without boundary terms) and the *boundary action* (with boundary terms). For the bulk action we are only interested in properties up to boundary terms. Thus the bulk action is used to find the equations of motion, on-shell supersymmetry transformations and conserved currents. For the derivation of the boundary conditions from an action we use the boundary action.

The article is organised as follows. In section 2 we review the boundary conditions which arise from the requirement of $N = 1$ superconformal invariance. We sketch the main results from [9] and [10]. Following this introductory section, in section 3 we treat the $N = 2$ sigma model on a Kähler manifold. This model has been extensively discussed in the literature because of its relevance to the description on D-branes on Calabi-Yau manifolds. We rederive the known results in a systematic fashion and find new results. We emphasize the differences between the present analysis and the previous studies of this type of boundary conditions. One of the interesting results we discuss is the full description of A-type branes with a B-field, as coisotropic submanifolds with f-structures on them. In the following section 4 we study the $N = 2$ superconformal boundary conditions for the bihermitean sigma model. From a technical point of view this problem is harder than the Kähler case partially due to the absence of a special system of coordinates such as the canonical complex coordinates in the Kähler case. However we derive the general results and give a partial interpretation of them in geometrical terms. It seems that a geometrical interpretation of these formal results requires a better mathematical insight into the problem than we have at present. Section 5 deals with the massive generalization of the models discussed in the previous sections. We are mainly concerned with the behaviour of the prepotential for certain supersymmetric solutions that admit a nice geometrical interpretation. Finally, in section 6 we give a summary of the paper with a discussion of the open problems and the relation of our result to previous results in the literature. There are rather detailed appendixes on $N = 1$ and $N = 2$ supersymmetry which establish our conventions and also point out some subtleties in the $N = 2$ formalism in the presence of boundaries. In the last three appendices some geometrical background is presented.

2. $N = 1$ review

In this section we review results on $N = 1$ superconformal boundary conditions for non-linear sigma models. This will allow us to introduce the notation and some relevant concepts. We closely follow ref's [9] and [10].

The starting point is a general $N = 1$ sigma model given by the bulk action

$$S = \int d^2\sigma\, d^2\theta\; D_+\Phi^\mu D_-\Phi^\nu (g_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)) \equiv \int d^2\sigma\, d^2\theta\; D_+\Phi^\mu D_-\Phi^\nu E_{\mu\nu}(\Phi) \tag{2.1}$$

where $E_{\mu\nu} \equiv g_{\mu\nu} + B_{\mu\nu}$ and we use a $N = 1$ superfield formalism (see appendix A for conventions). The model is defined over a riemannian manifold of dimension d with $g_{\mu\nu}$ being the riemannian metric[1] and $B_{\mu\nu}$, a general antisymmetric field. The theory is classically conformally invariant and thus there is no dimensionful parameter in the classical theory. With the assignment $d[\sigma] = 1$ and $d[\partial] = -1$, the canonical dimensions of the fields are $d[X] = 0$, $d[\psi] = 1/2$. A general ansatz[2] for the fermionic boundary conditions has the following form

$$\psi_-^\mu = \mathcal{R}^\mu(\psi_+, X) = R^\mu_{\ \nu}(X)\psi_+^\nu + R^\mu_{\ \nu\rho\sigma}(X)\psi_+^\nu\psi_+^\rho\psi_+^\sigma + \dots\,. \tag{2.2}$$

However, from dimensional analysis it may be seen that in the classical theory there are no higher fermion terms. Even for the most general ansatz where we allow $\mathcal{R}$ to depend on derivatives of the fields the only possible *local* dimension $-1/2$ term is the first term on the right hand side of (2.2). Other terms which are allowed by dimensional analysis have the form $\psi^k(\partial)^{-1/2(k-1)}$ and are thus nonlocal. In the quantum theory a mass scale is typically generated and thus higher fermion terms are allowed. It is easy to see that the problem may then be solved order by order in the fermions.

Since we are interested in the classical boundary conditions we start from the following fermionic boundary conditions

$$\psi_-^\mu = \eta R^\mu_{\ \nu}(X)\psi_+^\nu\,, \tag{2.3}$$

[1]Hereafter,by "riemannian", we shall mean "riemannian or pseudo riemannian".

[2]We need to assume differentiability to be able to do calculations. Thus within this classical approach we have to exclude such brane configurations where this property is lost (e.g., a brane ending on a brane).

which are the *most general local* fermionic boundary conditions in the classical theory. For the properties of R we introduce a terminology which generalizes the well-known concepts for D-branes in flat space-time. At a given point X, by going to special system of coordinates R can always be brought to the form

$$R = \begin{pmatrix} R^m_{\ n} & 0 \\ 0 & -\delta^i_{\ j} \end{pmatrix}. \tag{2.4}$$

Thus we may introduce a projector Q at this point ($Q^2 = Q$)

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & \delta^i_{\ j} \end{pmatrix}. \tag{2.5}$$

It follows that R, must satsify, in covariant language,

$$RQ = QR = -Q\,, \qquad Q^2 = Q\,. \tag{2.6}$$

A maximal projector Q with the property (2.6) will be called a Dirichlet projector, (justified below). If the rank of such a Q is $d-(p+1)$ we say that R corresponds to a Dp-brane. The projector which is complementary to Q is $\pi = I - Q$. We refer to π as a Neumann projector. The following properties of Q, π and R will be useful

$$\pi^2 = \pi\,, \qquad Q^2 = Q\,, \qquad r^2 = I\,, \qquad P\pi = \pi P = P\,, \qquad PQ = QP = 0 \tag{2.7}$$

where $r \equiv \pi - Q$ and $2P \equiv I + R$. In the absence of a B-field[3], the relation $R^2 = 1$ holds and this case corresponds to parity invariant boundary conditions.

We consider boundary conditions from the most general point of view, the off-shell supersymmetry transformations of the scalar mupliplet. Applying the off-shell supersymmetry transformations (A.6) to the ansatz (2.3) we obtain

$$\partial_{=}X^\mu - R^\mu_{\ \nu}\partial_{+\!\!+}X^\nu - 2i\eta P^\mu_{\ \nu}F^\nu_{+-} + 2iP^\sigma_{\ \rho}R^\mu_{\ \nu,\sigma}\psi^\rho_+\psi^\nu_+ = 0 \tag{2.8}$$

where $\epsilon^- = \eta\epsilon^+$ with $\eta^2 = 1$ in (A.6). Contracting with Q we get

$$Q^\mu_{\ \nu}\partial_0 X^\nu + 2iP^\rho_{\ \gamma}P^\nu_{\ \sigma}Q^\mu_{\ \nu,\rho}\psi^\sigma_+\psi^\gamma_+ = 0\,. \tag{2.9}$$

To understand this relation better let us jump ahead a bit and discuss the special case when $Q^\mu_{\ \nu}\partial_0 X^\nu = 0$ (or equivalently $\pi^\mu_{\ \nu}\partial_0 X^\nu = \partial_0 X^\mu$). Thus in (2.9) the two-fermion term should vanish by itself implying that

$$P^\mu_{\ \nu}P^\rho_{\ \sigma}Q^\lambda_{\ [\mu,\rho]} = 0\,. \tag{2.10}$$

Because $\pi P = P\pi = P$ we can rewrite the condition (2.10) in a completly equivalent form as follows

$$\pi^\mu_{\ \nu}\pi^\rho_{\ \sigma}Q^\lambda_{\ [\mu,\rho]} = 0 \tag{2.11}$$

[3]By B-field we shall mean the sum of the actual NS B-filed and the U(1) field strength.

which is the integrability condition for π. Due to this condition the Dp-brane can be interpreted as a maximal integral submanifold and $\partial_0 X^\mu$ as living in the tangent space of this submanifold, as was pointed out in [9].

It is important to realize that the above argument is independent of the details of the model (e.g., on-shell realization of supersymmetry transformation), as it is based only on the form of the ansatz (2.3) and the off-shell supersymmetry transformations for the scalar multiplet. Therefore this geometrical interpretation is still valid, e.g., for the massive sigma model (the Landau-Ginzburg model). However for the Landau-Ginzburg model the argument for the uniqueness of the ansatz (2.3) breaks down due to the presence of a dimensionful parameter.

Returning to the $N = 1$ conformal sigma model (2.1), the on-shell supersymmetry transformation together with ansatz (2.3) fixes the boundary conditions to be of the following form

$$\begin{cases} \psi_-^\mu = \eta R^\mu{}_\nu \psi_+^\nu \\ \partial_= X^\mu - R^\mu{}_\nu \partial_{\#} X^\nu + 2i \left(P^\sigma{}_\gamma \nabla_\sigma R^\mu{}_\nu + P^\mu{}_\rho g^{\rho\delta} H_{\delta\sigma\gamma} R^\sigma{}_\nu \right) \psi_+^\gamma \psi_+^\nu = 0\,. \end{cases} \tag{2.12}$$

Now we sketch the main steps in the derivation of the general $N = 1$ superconformal boundary conditions [10]. These will be relevant to what follows in that the requirement of $N = 1$ superconformal invariance is part of the $N = 2$ superconformal invariance. The components of the supercurrent and the stress tensor that we need may be taken to have the following expressions

$$G_+ = \psi_+^\mu \partial_{\#} X^\nu g_{\mu\nu} - \frac{i}{3} \psi_+^\mu \psi_+^\nu \psi_+^\rho H_{\mu\nu\rho}\,, \tag{2.13}$$

$$G_- = \psi_-^\mu \partial_= X^\nu g_{\mu\nu} + \frac{i}{3} \psi_-^\mu \psi_-^\nu \psi_-^\rho H_{\mu\nu\rho}\,, \tag{2.14}$$

$$T_{++} = \partial_{\#} X^\mu \partial_{\#} X^\nu g_{\mu\nu} + i \psi_+^\mu \nabla_+^{(+)} \psi_+^\nu g_{\mu\nu}\,, \tag{2.15}$$

$$T_{--} = \partial_= X^\mu \partial_= X^\nu g_{\mu\nu} + i \psi_-^\mu \nabla_-^{(-)} \psi_-^\nu g_{\mu\nu}\,, \tag{2.16}$$

where the covariant derivatives acting on the worldsheet fermions are defined by

$$\nabla_\pm^{(+)} \psi_+^\nu = \partial_{\substack{\#\\=}} \psi_+^\nu + \Gamma^{+\nu}_{\rho\sigma} \partial_{\substack{\#\\=}} X^\rho \psi_+^\sigma\,, \qquad \nabla_\pm^{(-)} \psi_-^\nu = \partial_{\substack{\#\\=}} \psi_-^\nu + \Gamma^{-\nu}_{\rho\sigma} \partial_{\substack{\#\\=}} X^\rho \psi_-^\sigma\,. \tag{2.17}$$

To ensure $N = 1$ superconformal invariance we have to impose the following conditions on the above currents on the boundary

$$T_{++} - T_{--} = 0\,, \qquad G_+ - \eta G_- = 0\,. \tag{2.18}$$

Classically these conditions make sense only on-shell since the conserved currents are defined modulo the equations of motion. Thus we should make use of the field equations in our analysis. Using the fermionic equations of motion,

$$g_{\mu\nu}(\psi_+^\mu \nabla_-^{(+)} \psi_+^\nu - \psi_-^\mu \nabla_+^{(-)} \psi_-^\nu) = 0\,, \tag{2.19}$$

we substitute the fermionic ansatz in the conformal condition and get

$$
\begin{aligned}
0 = T_{++} - T_{--} = {} & 2i\psi_+^\sigma \partial_0 \psi_+^\lambda \left[g_{\sigma\lambda} - R^\mu{}_\sigma g_{\mu\nu} R^\nu{}_\lambda\right] + \\
& + 2\partial_0 X^\delta \pi^\rho{}_\delta \Big[g_{\rho\nu}(\partial_{+\!\!\!+} X^\nu - \partial_= X^\nu) - 2B_{\rho\nu}\pi^\nu{}_\lambda \partial_0 X^\lambda + \\
& \qquad + i\Big(R^\mu{}_\gamma \Gamma_{\mu\rho\nu} R^\nu{}_\sigma - \Gamma_{\gamma\rho\sigma} - R^\mu{}_\sigma g_{\mu\nu} R^\nu{}_{\gamma,\rho} + \\
& \qquad + H_{\sigma\rho\gamma} + R^\mu{}_\sigma H_{\mu\rho\nu} R^\nu{}_\gamma \Big)\psi_+^\sigma \psi_+^\gamma \Big],
\end{aligned} \tag{2.20}
$$

where we assume that the string is confined to some of the directions (i.e., that there is a projector Q, $Q^2 = Q$ such that $Q^\mu{}_\nu \partial_0 X^\nu = 0$ or $\pi^\mu{}_\nu \partial_0 X^\nu = \partial_0 X^\mu$, $\pi \equiv I - Q$). In (2.20) we introduce an antisymmetric tensor $B_{\mu\nu}$ which a priori has nothing to do with H. The term which contains B vanishes identically in this expression and represents the ambiguity in the right hand side. We further stress that we do not assume any relation between Q and R (a condition $RQ = QR = -Q$ will arise from the supersymmetry requirements below). From the condition (2.20) we get the conformal boundary conditions

$$
\begin{cases}
\psi_-^\mu - \eta R^\mu{}_\nu \psi_+^\nu = 0\,, \\
\pi^\rho_\delta E_{\nu\rho}\pi^\nu_\lambda \partial_{+\!\!\!+} X^\lambda - \pi^\rho_\delta E_{\rho\nu}\pi^\nu_\lambda \partial_= X^\lambda - i\pi^\rho_\delta (R^\mu_\sigma g_{\mu\nu}\nabla_\rho R^\nu_\gamma - H_{\sigma\rho\gamma} - R^\mu_\sigma H_{\mu\rho\nu} R^\nu_\gamma)\psi_+^\sigma \psi_+^\gamma = 0\,, \\
Q^\mu{}_\lambda (\partial_= X^\lambda + \partial_{+\!\!\!+} X^\lambda) = 0\,,
\end{cases} \tag{2.21}
$$

together with the condition

$$
R^\mu{}_\sigma g_{\mu\nu} R^\nu{}_\rho = g_{\sigma\rho}\,. \tag{2.22}
$$

Substituting these relations into the second condition in (2.18), $G_+ - \eta G_- = 0$, we find the relations

$$
\begin{cases}
Q^\mu{}_\nu R^\nu{}_\rho = R^\mu{}_\nu Q^\nu{}_\rho = -Q^\mu{}_\rho \\
\pi^\rho{}_\delta E_{\nu\rho}\pi^\nu{}_\gamma = \pi^\rho{}_\delta E_{\rho\nu}\pi^\nu{}_\lambda R^\lambda{}_\gamma \\
\pi^\mu{}_\gamma \pi^\rho{}_\phi \nabla_{[\rho} Q^\delta{}_{\mu]} = 0 \\
\pi^\mu{}_\tau \pi^\nu{}_\sigma \pi^\rho{}_\gamma H_{\mu\nu\rho} = \frac{1}{2}\pi^\mu{}_\tau \pi^\nu{}_\sigma \pi^\rho{}_\gamma (\nabla_\mu B^{\mathcal{D}}_{\nu\rho} + \nabla_\nu B^{\mathcal{D}}_{\rho\mu} + \nabla_\rho B^{\mathcal{D}}_{\mu\nu})
\end{cases} \tag{2.23}
$$

where $B^{\mathcal{D}} \equiv \pi^t B \pi + Q^t B Q$. (Alternatively in the last line of (2.23) $B^{\mathcal{D}}$ may be replaced by $B^\pi \equiv \pi^t B \pi$.) If $H = dB$ then the last property in (2.23) would be trivially satisfied because of the integrability of π (the next to last condition). Using the properties (2.22) and (2.23) we see that the result is compatible with the supersymmetry algebra (i.e., we may rewrite the bosonic conditions in (2.21) in the form they have in (2.12)). In what follows we shall sometimes need

$$
R^\mu{}_\nu = r^\mu{}_\nu - 2g^{\mu\lambda} B^\pi_{\lambda\rho} P^\rho{}_\nu\,, \tag{2.24}
$$

which follows from (2.23).

All the above results can be derived differently, starting from an appropriate action which we now describe.

If we look for parity invariant boundary conditions (i.e., $R^2 = I$) we must start from an action without explicit boundary terms

$$
S = \int d^2\xi d^2\theta \; D_+ \Phi^\mu D_- \Phi^\nu g_{\mu\nu}(\Phi)\,. \tag{2.25}
$$

Requiring that the boundary field equations are satified as well as invariance under supersymmetry, including the boundary, we reproduce the $N = 1$ superconformal boundary conditions.

In the general case with non-zero B we have to use the following action with explicit boundary terms,

$$S = \int d^2\xi d^2\theta \; D_+\Phi^\mu D_-\Phi^\nu E_{\mu\nu}(\Phi) - \frac{i}{2}\int d^2\xi \; \partial_= \left(B_{\mu\nu}\psi_+^\mu\psi_+^\nu + B_{\mu\nu}\psi_-^\mu\psi_-^\nu\right). \tag{2.26}$$

In contrast to the current analysis we do not have to assume that $H = dB$ here since this is so by construction. For details of the derivation see [9] and [10].

The literature does not agree on the presence of the two-fermion terms in the bosonic boundary conditions. Therefore we would like to elaborate on this point. In the previous discussion we have outlined the proof that there is a one to one correspondence between the classical $N = 1$ superconformal boundary conditions of the sigma model and submanifolds with a B-field (the π-integrable r and the B-term in (2.24)). If we relax the requirements of $N = 1$ superconformal invariance to conformal invariance only, or to supersymmetric invariance only, the correspondence would no-longer be one to one. For example, if we adopt the boundary conditions $\psi_- = \eta R\psi_+$ and $\partial_= X = R\partial_{+\!\!+} X$, then the supersymmetry current conditions in (2.18) are satisfied (with $H = 0$)

$$G_+ - \eta G_- = \psi_+^\mu g_{\mu\nu}\partial_{+\!\!+} X^\nu - \eta\psi_-^\mu g_{\mu\nu}\partial_= X^\nu = 0 \tag{2.27}$$

provided that $R^t g R = g$. To interpret these boundary conditions as representing submanifolds of the target manifold, however, we have to add the inegrability conditions by hand. Further, unless extra requirement are imposed on R these solutions are not conformally invariant (i.e., $T_{++} - T_{--} \neq 0$). The extra requirements derived in this manner are much stronger than those previously discussed. Hence we see how different requiremens on the boundary conditions lead to the different results and interpretations. This partially explains the disagreement in the literature.

However if we look at *the* $N = 1$ *superconformal boundary conditions* (2.12) (together with the properties (2.22) and (2.23)) the two fermion terms *are* there in the general solution. It still makes sense, of course to ask under which circumstances the two-fermion term is absent in the boundary conditions. There is a simple theorem, for the parity invariant boundary conditions, which shows that the two-fermion term is absent in the bosonic boundary condition if and only if the submanifold is totally geodesic (for the details we refer to the appendix C). If the two-fermion term vanishes then the following property holds on the boundary

$$\delta X^\mu g_{\mu\nu}\partial_1 X^\nu = 0\,. \tag{2.28}$$

Thus we see that the property (2.28) for the parity invariant $N = 1$ superconformal boundary conditions is equivalent to the statement that the corresponding submanifold is totally geodesic. In fact the property of being totally geodesic has a very simple physical interpretation; once a geodesic starts along the submanifold it cannot escape the submanifold since the second fundamental form is identically zero. Therefore, classically, particles cannot escape from the D-brane in this case.

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3. $N=2$ sigma models on Kähler manifolds

We start our analysis from the relatively well studied example of an $N=2$ sigma model with action

$$S=\int d^2\sigma\, d^2\theta\ D_+\Phi^\mu D_-\Phi^\nu g_{\mu\nu}(\Phi)\,. \tag{3.1}$$

This action is written using $N=1$ superfield notation and is thus manifestly $N=1$ supersymmetric. For irreducible even dimensional manifolds the action (3.1) has an additional (nonmanifest) supersymmetry if and only if the manifold is Kähler [1]. The extra supersymmetry transformation is

$$\delta_2\Phi^\mu=\epsilon_2^\alpha D_\alpha\Phi^\nu\, J^\mu_\nu(\Phi) \tag{3.2}$$

where J^μ_ν is the covariantly constant complex structure of the manifold. The metric $g_{\mu\nu}$ should be hermitean with respect to this complex structure. The action (3.1) can be rewritten in a manifestly $N=2$ supersymmetric form as

$$S=\int d^2\sigma\, d^2\theta\, d^2\bar\theta\ K(\Phi,\bar\Phi) \tag{3.3}$$

where the superfields are now complex and K is a Kähler potential for the hermitean metric.

The boundary conditions for this model were first considered in [2]. Here we reproduce some of their results from a different point of view and complement them by new ones.

3.1 The $N=2$ algebra

A lot of information about the $N=2$ supersymmetric boundary conditions can be deduced by purely algebraic considerations. In components the manifest on-shell[4] supersymmetry transformations are

$$\begin{cases}\delta_1 X^\mu=-\left(\epsilon_1^+\psi_+^\mu+\epsilon_1^-\psi_-^\mu\right)\\ \delta_1\psi_+^\mu=-i\epsilon_1^+\partial_{+\!\!\!+}X^\mu+\epsilon_1^-\Gamma^\mu{}_{\nu\rho}\psi_-^\rho\psi_+^\nu\\ \delta_1\psi_-^\mu=-i\epsilon_1^-\partial_{=}X^\mu-\epsilon_1^+\Gamma^\mu{}_{\nu\rho}\psi_-^\rho\psi_+^\nu\end{cases} \tag{3.4}$$

and the nonmanifest tranformations (3.2) are

$$\begin{cases}\delta_2 X^\mu=\left(\epsilon_2^+\psi_+^\nu+\epsilon_2^-\psi_-^\nu\right)J^\mu_\nu\\ \delta_2\psi_+^\mu=-i\epsilon_2^+\partial_{+\!\!\!+}X^\nu J^\mu_\nu-\epsilon_2^-J^\mu_\sigma\Gamma^\sigma{}_{\nu\rho}\psi_-^\rho\psi_+^\nu-\epsilon_2^+J^\mu_{\nu,\rho}\psi_+^\rho\psi_+^\nu-\epsilon_2^-J^\mu_{\nu,\rho}\psi_+^\rho\psi_-^\nu\\ \delta_2\psi_-^\mu=-i\epsilon_2^-\partial_{=}X^\nu J^\mu_\nu+\epsilon_2^+J^\mu_\sigma\Gamma^\sigma{}_{\nu\rho}\psi_-^\rho\psi_+^\nu-\epsilon_2^+J^\mu_{\nu,\rho}\psi_-^\rho\psi_+^\nu-\epsilon_2^-J^\mu_{\nu,\rho}\psi_-^\rho\psi_-^\nu\,.\end{cases} \tag{3.5}$$

A supersymmetry transformation of the fermionic boundary conditions

$$\psi_-^\mu=\eta_1 R^\mu{}_\nu(X)\psi_+^\nu \tag{3.6}$$

yields

$$\delta_i\psi_-^\mu=\eta_1 R^\mu{}_{\nu,\rho}\delta_i X^\rho\psi_+^\nu+\eta_1 R^\mu{}_\nu\delta_i\psi_+^\nu\,,\qquad i=1,2\,. \tag{3.7}$$

This is the corresponding bosonic boundary conditions.

[4] The auxiliary (F-)field is integrated out.

The first supersymmetry variation (3.4) applied to (3.6) yields the expression discussed in the previous section

$$\partial_{=}X^{\mu} - R^{\mu}{}_{\nu}\partial_{\#}X^{\nu} + 2iP^{\rho}{}_{\gamma}\nabla_{\rho}R^{\mu}{}_{\nu}\psi_{+}^{\gamma}\psi_{+}^{\nu} = 0 \tag{3.8}$$

where $\epsilon_1^- = \eta_1\epsilon_1^+$ ($\eta_1^2 = 1$) and $2P^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + R^{\mu}{}_{\nu}$. The second supersymmetry variation (3.5) yields

$$\partial_{=}X^{\mu} + (\eta_1\eta_2)J^{\mu}{}_{\sigma}R^{\sigma}{}_{\nu}J^{\nu}{}_{\gamma}\partial_{\#}X^{\gamma} + i\left[(\eta_1\eta_2)J^{\mu}{}_{\sigma}\nabla_{\rho}R^{\sigma}{}_{\nu}J^{\rho}{}_{\gamma} + J^{\mu}{}_{\sigma}\nabla_{\rho}R^{\sigma}{}_{\nu}J^{\rho}{}_{\lambda}R^{\lambda}{}_{\gamma}\right]\psi_{+}^{\gamma}\psi_{+}^{\nu} = 0 \tag{3.9}$$

where we used that $\epsilon_2^+ = \eta_2\epsilon_2^-$ and the fact that $\nabla_{\rho}J^{\mu}{}_{\nu} = 0$.

Equations (3.8) and (3.9) should be equivalent. Comparing the X-part we get the following condition

$$J^{\mu}{}_{\gamma}R^{\gamma}{}_{\nu}J^{\nu}{}_{\sigma} = -(\eta_1\eta_2)R^{\mu}{}_{\sigma} \tag{3.10}$$

or equivalently

$$J^{\mu}{}_{\gamma}R^{\gamma}{}_{\nu} = (\eta_1\eta_2)R^{\mu}{}_{\gamma}J^{\gamma}{}_{\nu}\,. \tag{3.11}$$

Using (2.22), i.e. that $R^{\mu}{}_{\rho}g_{\mu\nu}R^{\nu}{}_{\sigma} = g_{\rho\sigma}$, this equation can be equivalently rewritten as follows

$$R^{\mu}{}_{\rho}J_{\mu\nu}R^{\nu}{}_{\sigma} = (\eta_1\eta_2)J_{\rho\sigma}\,. \tag{3.12}$$

The case $\eta_1\eta_2 = 1$ corresponds to the so called B-type conditions and $\eta_1\eta_2 = -1$ to the A-type conditions, as defined in [2].

Using (3.10) we rewrite the equation (3.9) as follows

$$\partial_{=}X^{\mu} - R^{\mu}{}_{\nu}\partial_{\#}X^{\nu} + 2i(\eta_1\eta_2)J^{\mu}{}_{\lambda}J^{\rho}{}_{\gamma}P^{\sigma}{}_{\rho}\nabla_{\sigma}R^{\lambda}{}_{\nu}\psi_{+}^{\gamma}\psi_{+}^{\nu} = 0\,. \tag{3.13}$$

Comparing the two-fermion terms in (3.8) and (3.13) we obtain

$$\left(P^{\rho}{}_{\gamma}\nabla_{\rho}R^{\mu}{}_{\nu} - (\eta_1\eta_2)J^{\mu}{}_{\lambda}J^{\rho}{}_{\gamma}P^{\sigma}{}_{\rho}\nabla_{\sigma}R^{\lambda}{}_{\nu}\right)\psi_{+}^{\gamma}\psi_{+}^{\nu} = 0\,. \tag{3.14}$$

Using (3.11) and the antisymmetry of the fermions, (3.14) may be rewritten as

$$P^{\rho}{}_{[\gamma|}\nabla_{\rho}R^{\mu}{}_{|\nu]} - J^{\rho}{}_{[\gamma|}P^{\sigma}{}_{\rho}\nabla_{\sigma}R^{\mu}{}_{\lambda}J^{\lambda}{}_{|\nu]} = 0\,. \tag{3.15}$$

Finally, introducing the projectors $\Omega_{\pm} = 1/2(I \pm iJ)$ the condition (3.15) takes the form

$$\Omega^{\rho}_{\pm[\gamma|}P^{\sigma}{}_{\rho}\nabla_{\sigma}R^{\mu}{}_{\lambda}\Omega^{\lambda}_{\pm|\nu]} = 0\,. \tag{3.16}$$

This condition does not imply that the two-fermion term vanishes. However it requires two-fermion term to have a form which is compatible with the U(1) R-symmetry (see discussion in subsection 3.2).

So far, except in (3.12), we have used the $N = 2$ supersymmetry algebra only. As the next step we combine the $N = 2$ algebraic results (3.10) and (3.15) with the $N = 1$ current analysis. Indeed, as will be shown later, this gives us the full information about the $N = 2$ superconformal boundary conditions.

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3.1.1 B-type

We first consider the B-type conditions and combine the algebraic requirements (3.10) and (3.14) for $\eta_1\eta_2 = 1$ with the results from the section 2. On matrix form, the condition (3.11) reads

$$JR = RJ\,. \tag{3.17}$$

Using the Neumann and Dirichlet projectors and their properties (2.7), it follows from (3.17) that

$$QJ\pi = \pi JQ = 0\,, \qquad \pi J = J\pi\,, \qquad QJ = JQ\,. \tag{3.18}$$

These expressions are completly equivalent to the statement that $J = rJr$. From (3.18) it may be shown that R corresponds to a D-brane with odd p (i.e., the world-volume of the brane is even dimensional). From equation (2.24) it follows that

$$J\left(\pi^t B\pi\right)J = \left(\pi^t B\pi\right). \tag{3.19}$$

The conditions (3.14) are automatically satisfied due to the integrability and the property (2.22).

It is useful to rewrite the boundary conditions in the canonical complex coordinates for the complex structure such that $J^i{}_j = i\delta^i{}_j$, $J^{\bar i}{}_{\bar j} = -i\delta^{\bar i}{}_{\bar j}$ and $J^i{}_{\bar j} = J^{\bar i}{}_j = 0$. The condition (3.19) implies that $B^\pi_{ij} = B^\pi_{\bar i\bar j} = 0$, where the fields are complexified in the standard fashion. In these coordinates (3.10) implies

$$R^i{}_{\bar j} = R^{\bar i}{}_j = 0\,, \qquad \nabla_\rho R^i{}_{\bar j} = \nabla_\rho R^{\bar i}{}_j = 0\,. \tag{3.20}$$

The conditions that follow from (3.14) take the form

$$P^s_l \nabla_s R^i{}_j = P^s_j \nabla_s R^i{}_l\,, \qquad P^{\bar s}_{\bar l} \nabla_{\bar s} R^{\bar i}{}_{\bar j} = P^{\bar s}_{\bar j} \nabla_{\bar s} R^{\bar j}{}_{\bar l}\,. \tag{3.21}$$

Using the relations (2.24) and (3.19) it may be seen that (3.21) is equivalent to the following expressions

$$\pi^s{}_l \pi^i{}_j \nabla_s Q^j{}_k = 0\,, \qquad \pi^{\bar s}{}_{\bar l} \pi^{\bar i}{}_{\bar j} \nabla_{\bar s} Q^{\bar j}{}_{\bar k} = 0 \tag{3.22}$$

In complex coordinates the integrability conditions for π (the third property in (2.23)) read

$$\begin{aligned} &\pi^j{}_{[k} \pi^l{}_{p]} \nabla_l Q^i{}_j = 0,\ \pi^j{}_k \pi^{\bar l}{}_{\bar p} \nabla_{\bar l} Q^i{}_j = 0\,, \\ &\pi^{\bar j}{}_{[\bar k} \pi^{\bar l}{}_{\bar p]} \nabla_{\bar l} Q^{\bar i}{}_{\bar j} = 0,\ \pi^s{}_k \pi^{\bar j}{}_{\bar l} \nabla_s Q^{\bar i}{}_{\bar j} = 0 \end{aligned} \tag{3.23}$$

and thus $\pi^i{}_j$, $\pi^{\bar i}_{\bar j}$ are independently integrable. Using (2.22) we see that (3.22) follows from (3.23). Indeed the condition (3.22) is weaker than inegrability of π.

Summarizing the above, in complex cordinates we have two sets of $d/2$ fermionic and $d/2$ bosonic boundary conditions. One set is

$$\begin{cases} \psi^i_- = \eta_1 R^i{}_j \psi^j_+ \\ \partial_= X^i - R^i{}_j \partial_{+\!\!+} X^j + 4iP^{\bar k}_{\bar l} P^i{}_j \nabla_{\bar k} Q^j{}_s \psi^s_+ \bar\psi^{\bar l}_+ = 0 \end{cases} \tag{3.24}$$

and the other set is obtained by interchanging the bared and unbared indices.

Geometrical aspects of the above results will be discussed in subsection 3.4.

3.1.2 A-type

We now turn to the A-type boundary conditions. In this case the condition (3.11) has the form

$$JR = -RJ \tag{3.25}$$

We begin with the case without B-field (i.e., $R^2 = 1$). Using the properties (2.7) and (3.25) it may be seen that

$$QJQ = 0\,, \qquad \pi J\pi = 0\,, \qquad rJr = -J\,. \tag{3.26}$$

From (3.26) it follows that rank of Q is $d/2$ and so $\mathrm{rank}(\pi) = \mathrm{rank}(Q)$. For this case the conditions (3.14) is completly equivalent to inegrability of π: Using the inegrability of π we may show that (3.14) is indeed true. Conversely, contracting (3.14) with $P^\gamma_{\ \phi}$ and $P^\nu_{\ \epsilon}$ and using the property (3.26) we recover the inegrability of π (for this case $\pi = P$). Moreover for this case of so called middle-dimensional branes we cannot introduce a B-field.

Next we consider the boundary conditions with a B-field. Now (3.25) together with (2.7) and (2.24) imply

$$QJQ = 0\,, \qquad QJB^\pi = 0 \tag{3.27}$$

where B^π is the field along the brane. In this case at the point we can always bring R to the following block diagonal form

$$R^\mu_{\ \nu} = \begin{pmatrix} K^\alpha_{\ \beta} & 0 & 0 \\ 0 & \delta^n_{\ m} & 0 \\ 0 & 0 & -\delta^i_{\ j} \end{pmatrix} \tag{3.28}$$

where (ij) are the Dirichlet directions, $(\alpha\beta)$ are the Neumann directions along which B^π is non zero and (nm) are the remaining Neumann directions. In (3.28) $(I \pm K)^\alpha_{\ \beta}$ is invertible in the $(\alpha\beta)$-subspace, i.e. $rank(K) = rank(B^\pi)$. Combining (3.27) and (3.28) with the property

$$R^\mu_{\ \rho} J_{\mu\nu} R^\nu_{\ \lambda} = -J_{\rho\lambda} \tag{3.29}$$

we find the following form for $J_{\mu\nu}$

$$J_{\mu\nu} = \begin{pmatrix} J_{\alpha\beta} & 0 & 0 \\ 0 & 0 & J_{in} \\ 0 & -J_{in} & 0 \end{pmatrix}. \tag{3.30}$$

Since $J_{\mu\nu}$ is a non-degenerate antisymmetric matrix we must have

$$\frac{1}{2}\left(\mathrm{rank}(\pi) - \mathrm{rank}(K) + \mathrm{rank}(Q)\right) = \mathrm{rank}(Q) \tag{3.31}$$

or alternatively

$$\mathrm{rank}(\pi) = \frac{1}{2}\left(d + \mathrm{rank}(B^\pi)\right). \tag{3.32}$$

By definition $\mathrm{rank}(\pi)$ is the dimension of the world-volume of the brane. Thus we find that the dimensionality of the A-type D-brane crucially depends on the rank of the U(1) field strength of the brane. Because $\mathrm{rank}(\pi^t J\pi) = \mathrm{rank}(B^\pi)$ the pull-back of $J_{\mu\nu}$ to the brane is degenerate unless we deal with space-filling brane. However there is a non-trivial relation between the geometry of the brane and the allowed B-field. Let us look at the details of this relationship for the case of space-filling branes ($\mathrm{rank}(B^\pi) = d$). For a space-filling brane the R-matrix can be written as follows

$$R = \frac{1}{I + \hat{B}}(I - \hat{B}) \tag{3.33}$$

where $\hat{B} = g^{-1}B$ and $\hat{B}$ is invertible. Comparing (3.33) with (3.25) we arrive at the relation

$$J\hat{B}J\hat{B} = -I \tag{3.34}$$

which in its turn implies that $\tilde{J}^\mu_\nu \equiv J^{\mu\lambda}B_{\lambda\nu}$ is an almost complex structure. The corresponding Nijenhuis tensor for $\tilde{J}$ is

$$\mathcal{N}^\mu_{\nu\rho}\left(\tilde{J}\right) = \tilde{J}^\gamma_\nu \nabla_{[\gamma}\tilde{J}^\mu_{\ \rho]} - \tilde{J}^\gamma_\rho \nabla_{[\gamma}\tilde{J}^\mu_{\ \nu]} = J^{\mu\lambda}\nabla_\lambda J_{\nu\rho} = 0\,, \tag{3.35}$$

where we used that J is covariantly constant and $dB = 0$. Thus the new almost complex structure $\tilde{J}$ is integrable. Using the definition of $\tilde{J}$ one finds

$$\tilde{J}^\mu_{\ \lambda} J_{\mu\rho} \tilde{J}^\rho_{\ \sigma} = -J_{\lambda\sigma}\,, \tag{3.36}$$

i.e. the Kähler form $J_{\mu\nu}$ is a (2,0)+(0,2) form with respect to the new complex structure $\tilde{J}$. Since the Kähler form $J_{\mu\nu}$ is nondegenerate we need the dimension of $\mathcal{M}$ to be multiple of 4. These are all requirements which follow from the superconformal invariance for the A-type space-filling brane.

The above structure is realized in some well-known situations. For example, assuming that $\tilde{J}$ is compatible with the metric (i.e., $\tilde{J}^t g \tilde{J} = g$) the relation (3.36) implies that $\{\tilde{J}, J\} = 0$. Thus there is one extra almost complex structure $\hat{B}$, ($\hat{B}^2 = -I$). This situation may be realized for 4k-dimesional manifolds, e.g. for hyperKähler manifolds. In this case the three complex structures $(J, \tilde{J}, \hat{B})$ can be shown to satisfy the standard SU(2) algebra.

In the non extreme cases ($0 < \mathrm{rank}(B^\pi) < d$) the pull back of $J_{\mu\nu}$ is degenerate on the brane. However due to the property (3.27) the complex structures J restricted to the tangent vectors satisfies the following equation

$$(J^\pi)^3 + J^\pi = 0 \tag{3.37}$$

where $J^\pi \equiv \pi J \pi$. Equation (3.37) is the generalization of the equation $J^2 = -I$ to the degenerate matrices. Analysing the structure of R in this case we further find that $\hat{B}^\pi \equiv \pi g^{-1} B \pi$ satisfies the condition

$$\left(\hat{B}^\pi J^\pi\right)^3 + \hat{B}^\pi J^\pi = 0\,. \tag{3.38}$$

The kernels of $B^\pi_{\mu\nu}$ and of the pull back of $J_{\mu\nu}$ coincide by construction. Thus, as in the discussion of (3.36), we see that $\mathrm{rank}(B^\pi)$ is a multiple of 4. We comment more on the geometrical interpretation these solution in subsection 3.4. When B^π is of maximal rank, equation (3.38) is equivalent to (3.34).

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As in the B-type case, it is useful to rewrite boundary conditions in the canonical complex coordinates. The relation (3.10) now implies

$$R^i{}_j = R^{\bar{i}}{}_{\bar{j}} = 0\,, \qquad \nabla_\rho R^i{}_j = \nabla_\rho R^{\bar{i}}{}_{\bar{j}} = 0\,, \tag{3.39}$$

and the integrability conditions for π become

$$\nabla_{[\bar{s}} R^i{}_{\bar{j}]} = 0\,, \qquad R^l{}_{[\bar{s}|} \nabla_l R^i{}_{|\bar{j}]} = 0\,, \qquad \nabla_{[s} R^{\bar{i}}{}_{j]} = 0\,, \qquad R^{\bar{l}}{}_{[s|} \nabla_{\bar{l}} R^{\bar{i}}{}_{|j]} = 0\,, \tag{3.40}$$

In complex cordinates the fermionic and bosonic boundary conditions thus read

$$\begin{cases} \psi^i_- = \eta_1 R^i{}_{\bar{j}} \bar{\psi}^{\bar{j}}_+ \\ \partial_= X^i - R^i{}_{\bar{j}} \partial_{+\!\!+} \bar{X}^{\bar{j}} + i(\nabla_s R^i{}_{\bar{j}} + R^{\bar{l}}{}_s \nabla_{\bar{l}} R^i{}_{\bar{j}}) \psi^s_+ \bar{\psi}^{\bar{j}}_+ = 0 \end{cases} \tag{3.41}$$

and the other set of boundary conditions is again obtained by interchanging the bared and unbared indices.

Geometrical aspects of these results are discussed in subsection 3.4.

3.2 Currents

Next we would like to rederive all the previous results in an alternative way: by imposing the boundary conditions on the appropriate currents.

We want to retain the classical $N = 2$ superconformal invariance in the presence of boundaries. Therefore the appropriate objects to study are the currents that correspond to supertranslations in (2,2) superspace. However it is instructive to look at the corresponding currents in the $N = 1$ formalism. This has the advantage that we are not confined to the use of complex coordinates. The currents are

$$T^-_{+\!\!+} = D_+\Phi^\mu \partial_{+\!\!+} \Phi^\nu g_{\mu\nu}\,, \qquad T^+_= = D_-\Phi^\mu \partial_= \Phi^\nu g_{\mu\nu} \tag{3.42}$$

$$\mathcal{J}_+ = D_+\Phi^\mu D_+\Phi^\nu J_{\mu\nu}\,, \qquad \mathcal{J}_- = D_-\Phi^\mu D_-\Phi^\nu J_{\mu\nu}\,. \tag{3.43}$$

Using that the manifold is Kähler together with the equations of motion we derive the following conservation laws

$$D_+ T^+_= = 0\,, \qquad D_- T^-_{+\!\!+} = 0\,, \qquad D_- J_+ = 0\,, \qquad D_+ J_- = 0\,. \tag{3.44}$$

The relevant components of the currents are

$$\begin{aligned} T_{++} &= -iD_+ T^-_{+\!\!+}| = \partial_{+\!\!+} X^\mu \partial_{+\!\!+} X^\nu g_{\mu\nu} + i\psi^\mu_+ \nabla_- \psi^\nu_+ g_{\mu\nu}\,, && (3.45)\\ T_{--} &= -iD_- T^+_=| = \partial_= X^\mu \partial_= X^\nu g_{\mu\nu} + i\psi^\mu_- \nabla_- \psi^\nu_- g_{\mu\nu} && (3.46)\\ G^1_+ &= T^-_{+\!\!+}| = \psi^\mu_+ \partial_{+\!\!+} X^\nu g_{\mu\nu}\,, \qquad G^1_- = T^+_=| = \psi^\mu_- \partial_= X^\nu g_{\mu\nu} && (3.47)\\ G^2_+ &= -\frac{i}{2} D_+ \mathcal{J}_+| = \psi^\mu_+ \partial_{+\!\!+} X^\nu J_{\mu\nu}\,, \qquad G^2_- = -\frac{i}{2} D_- \mathcal{J}_-| = \psi^\mu_- \partial_= X^\nu J_{\mu\nu} && (3.48)\\ J_+ &= \mathcal{J}_+| = \psi^\mu_+ \psi^\nu_+ J_{\mu\nu}\,, \qquad J_- = \mathcal{J}_-| = \psi^\mu_- \psi^\nu_- J_{\mu\nu}\,. && (3.49) \end{aligned}$$

In components the conservation laws acquire the following form

$$\partial_{\substack{+\!\!+\\=}} T_{\mp\mp} = 0\,, \qquad \partial_{\substack{+\!\!+\\=}} J_\mp = 0\,, \qquad \partial_{\substack{+\!\!+\\=}} G^a_\mp = 0\,, \quad a = 1, 2\,. \tag{3.50}$$

To ensure $N=2$ superconformal symmetry on the boundary we need to impose the following boundary conditions on the currents

$$T_{++} - T_{--} = 0\,, \qquad G^1_+ - \eta_1 G^1_- = 0\,, \qquad G^2_+ - \eta_2 G^2_- = 0\,, \qquad J_+ - (\eta_1\eta_2) J_- = 0\,. \tag{3.51}$$

In section 2 we have solved the first two conditions. Substituting the solutions (2.12) (where $H=0$) into the last two conditions in (3.51) gives

$$J_{\mu\nu} = (\eta_1\eta_2) R^{\rho}{}_{\mu} J_{\rho\sigma} R^{\sigma}{}_{\nu}\,. \tag{3.52}$$

Hence the current analysis coincides exactly with the previous results (cf. (3.12))

The conserved currents $J_\pm$ generate two R-rotations which act trivially on the bosonic fields but non-trivially on the fermions. Because of the boundary condition $J_+ - (\eta_1\eta_2)J_- = 0$ only one combination of these R-rotations survives as a symmetry in the presence of a boundary. Thus for the B-type we have the following R-symmetry

$$\begin{cases} \psi^\mu_+ \to \cos\alpha\ \psi^\mu_+ + \sin\alpha\ J^\mu{}_\nu \psi^\nu_+ \\ \psi^\mu_- \to \cos\alpha\ \psi^\mu_- + \sin\alpha\ J^\mu{}_\nu \psi^\nu_- \end{cases} \tag{3.53}$$

and for the A-type

$$\begin{cases} \psi^\mu_+ \to \cos\alpha\ \psi^\mu_+ + \sin\alpha\ J^\mu{}_\nu \psi^\nu_+ \\ \psi^\mu_- \to \cos\alpha\ \psi^\mu_- - \sin\alpha\ J^\mu{}_\nu \psi^\nu_-\,. \end{cases} \tag{3.54}$$

In complex coordinates these rotations take the familar form: $\psi^i_\pm \to e^{i\alpha}\psi^i_\pm$ for the B-type and $\psi^i_\pm \to e^{\pm i\alpha}\psi^i_\pm$ for the A-type. The boundary conditions (3.24) and (3.41) are invariant under these rotations respectively.

When using complex coordinates it is sometimes convenient to use a complexified version of the currents $G^a_\pm$ $(a=1,2)$:

$$\mathcal{G}_\pm = \frac{1}{2}\left(G^1_\pm + iG^2_\pm\right) = \psi^i_\pm \partial_{\substack{+\\=}} \bar{X}^{\bar{j}} g_{i\bar{j}}\,, \qquad \bar{\mathcal{G}}_\pm = \frac{1}{2}(G^1_\pm - iG^2_\pm) = \bar{\psi}^{\bar{i}}_\pm \partial_{\substack{+\\=}} X^j\,.g_{\bar{i}j} \tag{3.55}$$

Thus for the B-type case, the complex supercurrent boundary conditions have the form

$$\mathcal{G}_+ - \eta\mathcal{G}_- = 0\,, \qquad \bar{\mathcal{G}}_+ - \eta\bar{\mathcal{G}}_- = 0\,. \tag{3.56}$$

Making a R-rotation $\psi^i_\pm \to e^{\pm i\beta}\psi^i_\pm$ (which is not a symmetry of the model!) we arrive at the generalized B-type boundary conditions

$$\mathcal{G}_+ - \eta e^{-2i\beta}\mathcal{G}_- = 0\,, \qquad \bar{\mathcal{G}}_+ - \eta e^{2i\beta}\bar{\mathcal{G}}_- = 0 \tag{3.57}$$

(η can be absorbed into the phase if one wishes.) Thus these B-type conditions for the currents are solved by the conditions

$$\begin{cases} \psi^i_- - e^{-2i\beta} R^i{}_j \psi^j_+ = 0 \\ \partial_= X^i - R^i{}_j \partial_{+\!\!\!+} X^j + 4i P^{\bar{k}}_{\bar{l}} P^i{}_j \nabla_{\bar{k}} Q^j{}_s \psi^s_+ \bar{\psi}^{\bar{l}}_+ = 0 \end{cases} \tag{3.58}$$

where the bosonic boundary condition is related to the fermionic one through the $N=2$ supersymmetry transformations (B.8) with $\bar{\epsilon}^+ = e^{2i\beta}\bar{\epsilon}^-$. Although this argument follows the same lines as in [5], it should be noted is that the argument is applicable only after the form of the two-fermion terms in the bosonic boundary condition has been established.

The same argument can be applied to the A-type boundary conditions. The complex version of the boundary conditions for the supercurrent has the form

$$\mathcal{G}_+ - e^{-2i\beta}\bar{\mathcal{G}}_- = 0\,, \qquad \bar{\mathcal{G}}_+ - e^{2i\beta}\mathcal{G}_- = 0\,, \tag{3.59}$$

and is solved by

$$\begin{cases} \psi^i_- - e^{-2i\beta}R^i{}_{\bar{j}}\bar{\psi}^{\bar{j}}_+ = 0 \\ \partial_= X^i - R^i{}_{\bar{j}}\partial_{+\!\!+}\bar{X}^{\bar{j}} + i\left(\nabla_s R^i{}_{\bar{j}} + R^{\bar{l}}{}_s \nabla_{\bar{l}} R^i{}_{\bar{j}}\right)\psi^s_+\bar{\psi}^{\bar{j}}_+ = 0 \end{cases} \tag{3.60}$$

where the bosonic boundary condition is related to the fermionic one through the $N = 2$ supersymmetry transofrmations (B.8) with $\epsilon^+ = e^{2i\beta}\bar{\epsilon}^-$.

Thus we have established that the conditions (3.58) and (3.60) are the most general local solutions of the problem, in the two cases (within our framework and given our assumptions).

3.3 Actions

We now briefly review the derivation of the $N = 2$ superconformal boundary conditions starting from an action. There are essentially no new results in this subsection. However hopefully the present discussion will clarify some technical points related to the derivation of the supersymmetric boundary conditions from the action.

We start from the action

$$S = \int d^2\sigma \left[\partial_{+\!\!+} X^\mu \partial_= X^\nu E_{\mu\nu} + i\psi^\mu_+ \nabla_- \psi^\nu_+ g_{\mu\nu} + i\psi^\mu_- \nabla_+ \psi^\nu_- g_{\mu\nu} + \frac{1}{2}\psi^\mu_+\psi^\lambda_+\psi^\rho_-\psi^\nu_- \mathcal{R}_{\rho\nu\mu\lambda}\right], \tag{3.61}$$

where $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$ and it assumed that $dB = 0$. The action (3.61) is the action (2.26) written in components with the auxilary field integrated out. The boundary term in the field variation of S is given by

$$\begin{aligned} \delta S = i \int d\tau \Big[& (\delta\psi^\mu_+\psi^\nu_+ - \delta\psi^\mu_-\psi^\nu_-)g_{\mu\nu} + \delta X^\mu (i\partial_{+\!\!+} X^\nu E_{\nu\mu} - i\partial_= X^\nu E_{\mu\nu} + \\ & + (\psi^\nu_-\psi^\rho_- - \psi^\nu_+\psi^\rho_+)\Gamma_{\nu\mu\rho})\Big], \end{aligned} \tag{3.62}$$

and the supersymmetry variation of S is given by

$$\delta_1 S = \epsilon^-_1 \int d\tau \;\; [g_{\mu\nu}(\partial_{+\!\!+} X^\mu \psi^\nu_- - \eta_1 \partial_= X^\mu \psi^\nu_+) + 2\partial_0 X^\mu (\psi^\nu_- + \eta_1 \psi^\nu_+) B_{\mu\nu}] \tag{3.63}$$

with $\epsilon^+_1 = \eta_1 \epsilon^1_-$. Starting from the general fermionic ansatz (2.3) wee look for boundary conditions which set both varions (3.62) and (3.63) to zero. These were found in [9] and [10]. As discussed above, if the manifold is Kähler the action admits an extra supersymmetry. The variation of the action (3.61) under this extra supersymmetry is given by

$$\delta_2 S = \epsilon^-_2 \int d\tau \;\; \left[J_{\nu\mu}(\partial_{+\!\!+} X^\mu \psi^\nu_- - \eta_2 \partial_= X^\mu \psi^\nu_+) + 2\partial_0 X^\mu (\psi^\lambda_- + \eta_2 \psi^\lambda_+) J^\nu{}_\lambda B_{\mu\nu}\right] \tag{3.64}$$

with $\epsilon^+_2 = \eta_2 \epsilon^-_2$. Requiring the variation (3.64) to vanish we find exactly the same conditions (3.52) as in the previous section.

It is convenient to use complex coordinates. We may do this starting from the action (3.61) and rewriting it in complex form. However we must be extra careful if we start from the $N = 2$ action

$$S = \int d^2\sigma\, d^2\theta\, d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) \tag{3.65}$$

and reduce it to the component action in the presence of a boundary. The subtlety lies in the fermionic measure. Using (B.1) and (B.9) from appendix B along with the chirality conditions on Φ, we obtain the following expression for (3.65) reduced to $N = 1$

$$S = -\frac{1}{2} \int d^2\sigma\, d^2\theta\ \left(K_{i\bar{j}} D_{[-} \Phi^i D_{+]} \bar{\Phi}^{\bar{j}} - D_+ D_- K \right) \tag{3.66}$$

where $K_{i\bar{j}} \equiv \partial_i \bar{\partial}_{\bar{j}} K$. Comparing (3.66) to the standard $N = 1$ action we find an additional boundary term, $D_+ D_- K$. This term is also responsible for the fact that the Kähler gauge symmetry corresponding to the transformation

$$K(\Phi, \bar{\Phi}) \quad \rightarrow \quad K(\Phi, \bar{\Phi}) + h(\Phi) + \bar{h}(\bar{\Phi}) \tag{3.67}$$

is broken in the presence of the boundaries according to

$$\begin{aligned} \int d^2\sigma\, d^2\theta\, d^2\bar{\theta}\ \left[K(\Phi, \bar{\Phi}) + h(\Phi) + \bar{h}(\bar{\Phi}) \right] = & \int d^2\sigma\, d^2\theta\, d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \\ & + \frac{1}{4} \int d\tau \left(h_{,i} \partial_1 X^i + \bar{h}_{,\bar{i}} \partial_1 \bar{X}^{\bar{i}} \right). \end{aligned} \tag{3.68}$$

This is very much in analogy to the gauge symmetry for the antisymmetric 2-form gauge field B. The gauge symmetry $B \rightarrow B + d\Lambda$ is also broken by the boundary terms. However that symmetry can be restored by requiring an additional tranformation of a U(1) gauge field such that it compensates for the boundary terms. Looking at (3.68) it seems reasonable to assume that the boundary terms in (3.68) are canceled by appropriate transformations of the scalars transverse to the brane which couple to $\partial_1 X^\mu$ on the boundary.

3.4 Geometric interpretation of the boundary conditions

In this subsection we summarize our results from this section and translate them into statements about the geometry.

We have derived the *most general local* $N = 2$ superconformal boundary conditions in the classical theory. The $N = 1$ superconformal boundary conditions correspond to maximal integral submanifolds. The extra supersymmetry leads to further restrictions on the submanifolds of the Kähler target manifold, $\mathcal{M}$. The B-type boundary condition corresponds to a Kähler submanifold, D, which is invariant under the action of J

$$J\mathcal{T}_X(D) \subset \mathcal{T}_X(D)\,, \qquad J\mathcal{N}_X(D) \subset \mathcal{N}_X(D) \tag{3.69}$$

where $\mathcal{T}_X(D)$ is the tangent space of D at the point X and $\mathcal{N}_X(D)$ is the normal space at X. The property (3.69) follows automatically from (3.18). In fact J induces a complex structure on the submanifold which is integrable and covaraintly constant with respect to the induced connection. Thus the group of the tangent bundle of D can be reduced to U(k) where dim(D) $= 2k$. The B-type D-brane may have holomorphic (antiholomorphic) gauge fields on it.

The A-type boundary conditions correspond to the case when the symplectic (Kähler) structure resticted to the normal space is zero

$$J|_{\mathcal{N}_X(D)} = 0\,, \qquad \mathrm{rank}(J|_{\mathcal{T}_X(D)}) \leq \dim(\mathcal{T}_X(D))\,. \tag{3.70}$$

Thus A-type branes correspond to coisotropic manifolds with dimension

$$\dim(D) = \frac{1}{2}(d + \mathrm{rank}\left(J|_{\mathcal{T}_X(D)}\right)) \tag{3.71}$$

where $d = \dim(\mathcal{M})$ and $\mathrm{rank}(J|_{\mathcal{T}_X(D)}) = \mathrm{rank}(B^\pi)$ which should be muptiple of 4. B^π is the field strength of the U(1) field on the brane. The case of zero B^π (i.e., $J|_{\mathcal{T}_X(D)} = 0$) would correspond to a lagrangian submanifold allowing only flat gauge fields. The opposite case of maximal rank of B^π correspond to the space-filling brane which can be realized for $d = 4k$. To have an A-type space-filling brane there must be extra geometrical structure on the manifold $\mathcal{M}$. In the generic situation there is an extra complex structure $\tilde{J}$ with the property (3.36). This situation is realized in, e.g., hyperKähler geometry.

In the generic situation $0 < \mathrm{rank}(J|_{\mathcal{T}_X(D)}) < \dim(\mathcal{T}_X(D))$ we have the following decomposition of the tangent space of a D-brane

$$\mathcal{T}_X(D) = J\mathcal{N}_X(D) \oplus \hat{\mathcal{T}}(D)\,, \qquad J\mathcal{T}_X(D) = \mathcal{N}_X(D) \oplus \hat{\mathcal{T}}(D)\,. \tag{3.72}$$

We see that the space $\hat{\mathcal{T}}(D)$ is invariant under the action of J, that is $J\hat{\mathcal{T}}(D) = \hat{\mathcal{T}}(D)$. $J\mathcal{N}_X(D)$ is the kernel of $J_{\mu\nu}$ restricted to $\mathcal{T}_X(D)$ (it is also the kernel of B^π) and therefore this space is integrable. In the mathematical literature (see, e.g. [12]) submanifolds of this type are sometimes called generic CR submanifolds. The complex structures J restricted to the tangent vectors of D gives rise to an f-structure[5] on the submanifold, see equation (3.37). In fact, in this case the brane submanifold has two different f-structures, J^π and $\hat{B}^\pi J^\pi$ with properties (3.37) and (3.38). These f-structures restricted to $\hat{\mathcal{T}}(D)$ become invertible. The presence of the f-structure allows the reduction of the tangent bundle group of D to $\mathrm{U}(2r) \times O(d/2 - 2r)$ where $4r = \mathrm{rank}(J^\pi)$. Previously similar observations about the non-lagrangian A-type branes have been made in [11].

Let us point out a pecularity related to the derivation of the A-type boundary conditions without B-field. A careful look at the previous derivation reveals an essential difference between A- and B-types. Starting from the parity invariant ansatz (3.6) with $R^2 = I$ and using the $N = 2$ algebra only would give us two results for the A-type (i.e., $\eta_1\eta_2 = -1$);

$$RJR = -J\,, \qquad P^\mu_{\ [\rho} P^\nu_{\ \sigma]} \nabla_\mu Q^\lambda_{\ \nu} = 0 \tag{3.73}$$

where we recall that $2P = I + R$ and $2Q = I - R$. (The derivation is found in subsection 3.1.) For a geometrical interpretation, (3.73) is sufficient. The second relation says that we deal with a submanifold and the first realtion says that J maps tangent vectors to normal vectors and vice versa

$$J\mathcal{T}_X(D) \subset \mathcal{N}_X(D)\,, \qquad J\mathcal{N}_X(D) \subset \mathcal{T}_X(D)\,. \tag{3.74}$$

[5]For the definitions and the basic properties we refer to the appendix C

Thus we have a lagrangian submanifold. Note that this is a purely algebraic result. However, to uncover the proper geometrical interpretation for the B-type we have to use the current conditions (3.51). The condition (3.15) is too week to imply inegrability of π and thus we do not obtain a geometrical interpretation from the $N = 2$ algebra alone. This important difference between the A- and B-types is seen in the full analysis only.

Further if we want to satisfy property (2.28) for the $N = 2$ superconformal boundary conditions we find totally geodesic Kähler and lagrangian submanifolds respectively. Unlike in the $N = 1$ case, this theorem would now hold even in the presence of gauge fields for the B-type boundary conditions.

4. $N = 2$ sigma model on bihermitean manifolds

We now consider the $N = 1$ superfield bulk action for the real scalar superfields Φ^μ

$$S = \int d^2\sigma\, d^2\theta\; D_+\Phi^\mu D_-\Phi^\nu (g_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi))\,, \tag{4.1}$$

where we assume that $H \equiv dB \neq 0$. This action is manifestly supersymmetric under one supersymmetry because of its $N = 1$ superfield form. Further (4.1) admits an additional nonmanifest supersymmetry of the form

$$\delta_2\Phi^\mu = \epsilon_2^+ D_+\Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon_2^- D_-\Phi^\nu J^\mu_{-\nu}(\Phi) \tag{4.2}$$

where $D_\pm\epsilon_2^\mp = 0$ and $J^\mu_{\pm\nu}$ are the two complex structures [3]. The metric $g_{\mu\nu}$ has to be hermitean with respect to both complex structure (hence the name "bihermitean"). Each of these complex structures is covariantly constant, with respect to different connections however,

$$\nabla^{(\pm)}_\rho J^\mu_{\pm\nu} \equiv J^\mu_{\pm\nu,\rho} + \Gamma^{\pm\mu}_{\rho\sigma} J^\sigma_{\pm\nu} - \Gamma^{\pm\sigma}_{\rho\nu} J^\mu_{\pm\sigma} = 0\,, \tag{4.3}$$

where we defined the two affine connections

$$\Gamma^{\pm\mu}_{\rho\nu} = \Gamma^\mu{}_{\rho\nu} \pm g^{\mu\sigma} H_{\sigma\rho\nu}\,. \tag{4.4}$$

A few more formulae are useful. The inegrability of $J_\pm$ together with (4.3) lead to the following relation

$$H_{\delta\nu\lambda} = J^\sigma_{\pm\delta} J^\rho_{\pm\nu} H_{\sigma\rho\lambda} + J^\sigma_{\pm\lambda} J^\rho_{\pm\delta} H_{\sigma\rho\nu} + J^\sigma_{\pm\nu} J^\rho_{\pm\lambda} H_{\sigma\rho\delta}\,. \tag{4.5}$$

As another consequence of the constancy of the complex structures we may express the torsion H in terms of the complex structures $J_\pm$, [3]

$$H_{\mu\nu\rho} = -J^\lambda_{+\mu} J^\sigma_{+\nu} J^\gamma_{+\rho} (dJ_+)_{\lambda\sigma\gamma} = J^\lambda_{-\mu} J^\sigma_{-\nu} J^\gamma_{-\rho} (dJ_-)_{\lambda\sigma\gamma} \tag{4.6}$$

where

$$(dJ_\pm)_{\lambda\sigma\gamma} = \frac{1}{2}(\nabla_\lambda J_{\pm\sigma\gamma} + \nabla_\sigma J_{\pm\gamma\lambda} + \nabla_\gamma J_{\pm\lambda\sigma}). \tag{4.7}$$

The supersymmetry algebra is the usual one. As long as the two complex structures commute, the supersymmetry algebra closes off-shell and the model can be formulated in $(2,2)$

superspace. Commuting complex structures ($[J_-, J_+] = 0$) is equivalent to the existence of a hermitean locally riemannian product manifold [12] since then $\Pi = J_- J_+$ is an integrable almost product structure. In this case, which is very special from the geometrical point of view, the action (4.1) may be rewritten in a manifestly $N = 2$ supersymmetric form as

$$S = \int d^2\sigma\, d^2\theta\, d^2\bar{\theta}\; K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}) \tag{4.8}$$

where Φ and Λ are chiral and twisted chiral multiplets respectively. In the case of non-commuting complex structures the supersymmetry algebra closes only on-shell, hence the algebra is model dependent and a manifest supersymmetric formulation will require introduction of additional auxiliary fields. The construction of a manifest off-shell supersymmetric version of this model has been investigated in [13]–[17].

The main examples of this type of the geometry are given by WZW models. It is also known that any even dimensional group allows for a $N = 2$ super Kac-Moody symmetry [18, 19]. However for the WZW models the metric is never Kähler and $H \neq 0$. On these even dimensional group manifolds we find the geometry discussed above realized.

In what follows we derive formal results for the general case of a (2,2) sigma model with torsion. However due to lack of proper understanding of the underlying geometry it is difficult to interpret these results in geometrical terms except for special cases, e.g. for commuting complex structures.

4.1 The $N = 2$ algebra

As in the Kähler case a lot of information can be obtained from algebraic considerations. In components the manifest on-shell supersymmetry transformations are

$$\begin{cases} \delta_1 X^\mu = -(\epsilon_1^+ \psi_+^\mu + \epsilon_1^- \psi_-^\mu) \\ \delta_1 \psi_+^\mu = -i\epsilon_1^+ \partial_{+\!\!+} X^\mu + \epsilon_1^- \Gamma^{-\mu}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu \\ \delta_1 \psi_-^\mu = -i\epsilon_1^- \partial_{=} X^\mu - \epsilon_1^+ \Gamma^{-\mu}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu \end{cases} \tag{4.9}$$

and the nonmanifest supersymmetry transformations (4.2) are

$$\begin{cases} \delta_2 X^\mu = \epsilon_2^+ \psi_+^\nu J^\mu_{+\nu} + \epsilon_2^- \psi_-^\nu J^\mu_{-\nu} \\ \delta_2 \psi_+^\mu = -i\epsilon_2^+ \partial_{+\!\!+} X^\nu J^\mu_{+\nu} - \epsilon_2^- J^\mu_{-\sigma} \Gamma^{-\sigma}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu + \epsilon_2^+ J^\mu_{+\nu,\rho} \psi_+^\nu \psi_+^\rho + \epsilon_2^- J^\mu_{-\nu,\rho} \psi_-^\nu \psi_+^\rho \\ \delta_2 \psi_-^\mu = -i\epsilon_2^- \partial_{=} X^\nu J^\mu_{-\nu} + \epsilon_2^+ J^\mu_{+\sigma} \Gamma^{-\sigma}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu + \epsilon_2^+ J^\mu_{+\nu,\rho} \psi_+^\nu \psi_-^\rho + \epsilon_2^- J^\mu_{-\nu,\rho} \psi_-^\nu \psi_-^\rho\,. \end{cases} \tag{4.10}$$

As before, we start from the fermionic ansatz (3.6) and apply both supersymmetry transformations, (4.9) and (4.10). The result of the first transformation is

$$\partial_= X^\mu - R^\mu_{\ \nu} \partial_{+\!\!+} X^\nu + 2i(P^\sigma_{\ \gamma} \nabla_\sigma R^\mu_{\ \nu} + P^\mu_{\ \rho} g^{\rho\delta} H_{\delta\sigma\gamma} R^\sigma_{\ \nu}) \psi_+^\gamma \psi_+^\nu = 0 \tag{4.11}$$

where $\epsilon_1^+ = \eta_1 \epsilon_1^-$. The second supersymmetry gives

$$\begin{aligned} &\partial_= X^\mu + (\eta_1\eta_2) J^\mu_{-\lambda} R^\lambda_{\ \sigma} J^\sigma_{+\nu} \partial_{+\!\!+} X^\nu + i \Big[(\eta_1\eta_2) J^\mu_{-\lambda} \nabla^{(-)}_\rho R^\lambda_{\ \nu} J^\rho_{+\gamma} + \\ &+ (\eta_1\eta_2) J^\mu_{-\lambda} R^\lambda_{\ \sigma} J^\sigma_{+\rho} H^\rho_{\ \nu\gamma} + J^\mu_{-\lambda} \nabla^{(+)}_\rho R^\lambda_{\ \nu} J^\rho_{-\sigma} R^\sigma_{\ \gamma} - H^\mu_{\ \rho\sigma} R^\sigma_{\ \gamma} R^\rho_{\ \nu} \Big] \psi_+^\gamma \psi_+^\nu = 0 \end{aligned} \tag{4.12}$$

where $\epsilon_2^+ = \eta_2 \epsilon_2^-$ and we have used the property (4.3).

The boundary conditions (4.11) and (4.12) should be equivalent. Starting from the X-part we get the condition

$$(\eta_1\eta_2) J^{\mu}_{-\lambda} R^{\lambda}{}_{\sigma} J^{\sigma}_{+\nu} = -R^{\mu}{}_{\nu} \tag{4.13}$$

or equivalently

$$J^{\mu}_{-\nu} R^{\nu}{}_{\lambda} = (\eta_1\eta_2) R^{\mu}{}_{\nu} J^{\nu}_{+\lambda}\,. \tag{4.14}$$

Using $R^{\mu}{}_{\sigma} g_{\mu\nu} R^{\nu}{}_{\rho} = g_{\sigma\rho}$ we rewrite (4.14) as

$$R^{\mu}{}_{\sigma} J_{-\mu\nu} R^{\nu}{}_{\rho} = (\eta_1\eta_2) J_{+\sigma\rho}\,. \tag{4.15}$$

In analogy with the Kähler case we might call the case $\eta_1\eta_2 = 1$ a B-type and $\eta_1\eta_2 = -1$ an A-type condition. Using the property (4.14) the equation (4.12) is rewritten as

$$\begin{aligned}\partial_{=} X^{\mu} - R^{\mu}{}_{\nu}\partial_{+\!\!+} X^{\nu} + i\Big[&(\eta_1\eta_2) J^{\mu}_{-\lambda} \nabla^{(-)}_{\rho} R^{\lambda}{}_{\nu} J^{\rho}_{+\gamma} + \\ &+(\eta_1\eta_2) J^{\mu}_{-\lambda} \nabla^{(+)}_{\rho} R^{\lambda}{}_{\nu} R^{\rho}{}_{\sigma} J^{\sigma}_{+\gamma} - R^{\mu}{}_{\lambda} H^{\lambda}{}_{\nu\gamma} - H^{\mu}{}_{\rho\sigma} R^{\sigma}{}_{\gamma} R^{\rho}{}_{\nu}\Big]\psi^{\gamma}_{+}\psi^{\nu}_{+} = 0\,.\end{aligned} \tag{4.16}$$

Using (4.5) we further rewrite (4.16) as

$$\partial_{=} X^{\mu} - R^{\mu}{}_{\nu}\partial_{+\!\!+} X^{\nu} + 2i J^{\sigma}_{+\gamma} J^{\lambda}_{+\nu}\left(P^{\rho}{}_{\sigma}\nabla_{\rho} R^{\mu}{}_{\lambda} + P^{\mu}{}_{\phi} H^{\phi}{}_{\rho\sigma} R^{\rho}{}_{\lambda}\right)\psi^{\gamma}_{+}\psi^{\nu}_{+} = 0\,. \tag{4.17}$$

Comparing the two-fermion terms of (4.11) and (4.17) we get

$$P^{\sigma}{}_{[\gamma|}\nabla_{\sigma} R^{\mu}{}_{|\nu]} + P^{\mu}{}_{\rho} H^{\rho}{}_{\sigma[\gamma} R^{\sigma}{}_{\nu]} - J^{\sigma}_{+[\gamma} J^{\lambda}_{+\nu]}\left(P^{\rho}{}_{\sigma}\nabla_{\rho} R^{\mu}{}_{\lambda} + P^{\mu}{}_{\phi} H^{\phi}{}_{\rho\sigma} R^{\rho}{}_{\lambda}\right) = 0. \tag{4.18}$$

Using the projectors $\Omega^{+}_{\pm} = 1/2(I \pm iJ_{+})$ we rewrite the above condition as

$$\Omega^{+\sigma}_{\pm[\gamma}\Omega^{+\lambda}_{\pm\nu]}\left(P^{\rho}{}_{\sigma}\nabla_{\rho} R^{\mu}{}_{\lambda} + P^{\mu}{}_{\phi} H^{\phi}{}_{\rho\sigma} R^{\rho}{}_{\lambda}\right) = 0\,. \tag{4.19}$$

This condition is very similar to the corresponding condition for the Kähler case, (3.16). As before the condition (4.19) does not imply that the two-fermion term vanishes. However it requires the two-fermion term to have a form which is compatible with an appropriate U(1) R-symmetry.

4.2 Currents

Alternatively we derive the superconformal boundary conditions by imposing conditions on the conserved currents. In the form we need them, the currents are

$$T^{-}_{+\!\!+} = D_{+}\Phi^{\mu}\partial_{+\!\!+}\Phi^{\nu} g_{\mu\nu} - \frac{i}{3} D_{+}\Phi^{\mu} D_{+}\Phi^{\nu} D_{+}\Phi^{\rho} H_{\mu\nu\rho}\,, \tag{4.20}$$

$$T^{+}_{=} = D_{-}\Phi^{\mu}\partial_{=}\Phi^{\nu} g_{\mu\nu} + \frac{i}{3} D_{-}\Phi^{\mu} D_{-}\Phi^{\nu} D_{-}\Phi^{\rho} H_{\mu\nu\rho}\,, \tag{4.21}$$

$$\mathcal{J}_{+} = D_{+}\Phi^{\mu} D_{+}\Phi^{\nu} J_{+\mu\nu}\,, \tag{4.22}$$

$$\mathcal{J}_{-} = D_{-}\Phi^{\mu} D_{-}\Phi^{\nu} J_{-\mu\nu}\,. \tag{4.23}$$

$T^{-}_{\mp}$ and $T^{+}_{=}$ correspond to $N=1$ supercurrent and they were derived for the general case of the $N=1$ sigma model, e.g. in the appendix of [9]. The currents $\mathcal{J}_+$ and $\mathcal{J}_-$ generate the additional supersymmetry transformation (4.2). Using the equations of motion and the properties (4.3) together with the fact the $\nabla^{(\pm)}_\rho g_{\mu\nu}=0$ we find that the currents (4.20)–(4.23) are indeed conserved

$$D_+ T^+_{=} = 0\,, \qquad D_- T^-_{\mp} = 0\,, \qquad D_- \mathcal{J}_+ = 0\,, \qquad D_+ \mathcal{J}_- = 0\,. \tag{4.24}$$

The components of the currents (4.20)–(4.23) are related to the (2,2) currents ($T_{\pm\pm}$, $G^1_\pm$, $G^2_\pm$, $J_\pm$) as follows

$$\begin{aligned}
T_{++} &= -iD_+T^-_{\mp}| = \partial_{\mp} X^\mu \partial_{\mp} X^\nu g_{\mu\nu} + i\psi^\mu_+ \nabla^{(+)}_+ \psi^\nu_+ g_{\mu\nu}\,, && (4.25)\\
T_{--} &= -iD_-T^+_{=}| = \partial_{=} X^\mu \partial_{=} X^\nu g_{\mu\nu} + i\psi^\mu_- \nabla^{(-)}_- \psi^\nu_- g_{\mu\nu}\,, && (4.26)\\
G^1_+ &= T^-_{\mp}| = \psi^\mu_+ \partial_{\mp} X^\nu g_{\mu\nu} - \frac{i}{3}\psi^\mu_+\psi^\nu_+\psi^\rho_+ H_{\mu\nu\rho}\,, && (4.27)\\
G^1_- &= T^+_{=}| = \psi^\mu_- \partial_{=} X^\nu g_{\mu\nu} + \frac{i}{3}\psi^\mu_-\psi^\nu_-\psi^\rho_- H_{\mu\nu\rho}\,, && (4.28)\\
G^2_+ &= \frac{i}{2}D_+\mathcal{J}_+| = \psi^\mu_+ \partial_{\mp} X^\nu J_{+\mu\nu} + \frac{i}{3}\psi^\mu_+\psi^\nu_+\psi^\rho_+ J^\lambda_{+\mu} J^\sigma_{+\nu} J^\gamma_{+\rho} H_{\lambda\sigma\gamma} && (4.29)\\
G^2_- &= \frac{i}{2}D_-\mathcal{J}_-| = \psi^\mu_- \partial_{=} X^\nu J_{-\mu\nu} - \frac{i}{3}\psi^\mu_-\psi^\nu_-\psi^\rho_- J^\lambda_{-\mu} J^\sigma_{-\nu} J^\gamma_{-\rho} H_{\lambda\sigma\gamma} && (4.30)\\
J_+ &= \mathcal{J}_+| = \psi^\mu_+\psi^\nu_+ J_{+\mu\nu}\,, \qquad J_- = \mathcal{J}_-| = \psi^\mu_-\psi^\nu_- J_{-\mu\nu} && (4.31)
\end{aligned}$$

where the covariant derivatives acting on the worldsheet fermions are defined by

$$\nabla^{(+)}_\pm \psi^\nu_+ = \partial_{\frac{\mp}{=}} \psi^\nu_+ + \Gamma^{+\nu}_{\rho\sigma} \partial_{\frac{\mp}{=}} X^\rho \psi^\sigma_+\,, \qquad \nabla^{(-)}_\pm \psi^\nu_- = \partial_{\frac{\mp}{=}} \psi^\nu_- + \Gamma^{-\nu}_{\rho\sigma} \partial_{\frac{\mp}{=}} X^\rho \psi^\sigma_-\,. \tag{4.32}$$

To ensure $N=2$ superconformal symmetry on the boundary we impose the following conditions on the currents (4.25)–(4.31),

$$T_{++} - T_{--} = 0\,, \qquad G^1_+ - \eta_1 G^1_- = 0\,, \qquad G^2_+ - \eta_2 G^2_- = 0\,, \qquad J_+ - (\eta_1\eta_2) J_- = 0\,. \tag{4.33}$$

The two first conditions were solved completly in [10]. Using these results it is straightforward to find the content of the remaining two conditions. In total, the conditions from [10] has to be supplemented by the single condition

$$R^\rho{}_\mu J_{-\rho\sigma} R^\sigma{}_\nu = (\eta_1\eta_2) J_{+\mu\nu}\,, \tag{4.34}$$

which agrees with the algebraic results from subsection 4.1.

The conserved currents $J_\pm$ generate two R-rotations which act trivially on the bosonic fields but non-trivially on the fermions. Because of the boundary condition $J_+ - (\eta_1\eta_2)J_- = 0$ only one combination of these R-rotations survives as a symmetry in the presence of a boundary. Thus for $(\eta_1\eta_2)=1$ we have the following R-symmetry

$$\left\{ \begin{array}{l} \psi^\mu_+ \to \cos\alpha\ \ \psi^\mu_+ + \sin\alpha\ J^\mu_{+\nu}\psi^\nu_+ \\ \psi^\mu_- \to \cos\alpha\ \psi^\mu_- + \sin\alpha\ J^\mu_{-\nu}\psi^\nu_- \end{array} \right. \tag{4.35}$$

and for $(\eta_1\eta_2) = -1$

$$\begin{cases} \psi_+^\mu \to \cos\alpha\ \psi_+^\mu + \sin\alpha\ J_{+\nu}^\mu \psi_+^\nu \\ \psi_-^\mu \to \cos\alpha\ \psi_-^\mu - \sin\alpha\ J_{-\nu}^\mu \psi_-^\nu \end{cases} \tag{4.36}$$

Unlike the Kähler case these rotations do not simplify in complex form. We can choose the complex coordiates with respect to J_+ and then the rotations of ψ_+ will take the simple form: $\psi_+^\mu \to e^{i\alpha}\psi_+^\mu$. However in these coordinates the rotations of ψ_-^μ do not have a nice form except in the special case of commuting complex structures. Using the properties (4.14) and (4.19) it is easily checked that the form of the boundary conditions (3.6) and (4.11) is invariant under a combination of the respective U(1) rotations.

We face the same problem as above if we try to complexify the supersymmetry currents $G_\pm^i$, $i = 1, 2$. We cannot complexify G_+^i and G_-^i at the same time unless the complex structures commute. However we may repeat the same manipulations as for the Kähler case in subsection 3.2. As an example, we take $(\eta_1\eta_2) = 1$. Making a R-rotation (4.36) (which is not a symmetry of the model!) we arrive at the generalized $(\eta_1\eta_2) = 1$ boundary conditions

$$\begin{pmatrix} G_+^1 \\ G_+^2 \end{pmatrix} = \eta \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} G_-^1 \\ G_-^2 \end{pmatrix} \tag{4.37}$$

where we used the property (4.5). Similar manipulations may be done for the generalized $(\eta_1\eta_2) = -1$ boundary conditions.

4.3 Action

We derive the $N = 2$ superconformal boundary conditions starting from the appropriate action as in [10]. Here we sketch the main steps in the derivation of the full set of boundary conditions from the action: The correct action has the form

$$S = \int d^2\xi d^2\theta\ D_+\Phi^\mu D_-\Phi^\nu E_{\mu\nu}(\Phi) - \frac{i}{2}\int d^2\xi\ \partial_= \left(B_{\mu\nu}\psi_+^\mu\psi_+^\nu + B_{\mu\nu}\psi_-^\mu\psi_-^\nu\right). \tag{4.38}$$

or in component

$$S = \int d^2\xi\ \left[\partial_{+\!\!+} X^\mu \partial_= X^\nu E_{\mu\nu} + i\psi_+^\mu \nabla_-^{(+)}\psi_+^\nu g_{\mu\nu} + i\psi_-^\mu \nabla_+^{(-)}\psi_-^\nu g_{\mu\nu} + \frac{1}{2}\psi_+^\lambda\psi_+^\sigma\psi_-^\rho\psi_-^\gamma \mathcal{R}^-_{\rho\gamma\lambda\sigma}\right] \tag{4.39}$$

where we have integrated out the auxilary field. In (4.39) the curvature is defined as follows

$$\mathcal{R}^{\pm\mu}_{\ \ \sigma\rho\lambda} = \Gamma^{\pm\mu}_{\ \lambda\sigma,\rho} - \Gamma^{\pm\mu}_{\ \rho\sigma,\lambda} + \Gamma^{\pm\mu}_{\ \rho\gamma}\Gamma^{\pm\gamma}_{\ \lambda\sigma} - \Gamma^{\pm\mu}_{\ \lambda\gamma}\Gamma^{\pm\gamma}_{\ \rho\sigma} \tag{4.40}$$

with $\Gamma^\pm$ given by (4.4) and $\nabla_\mp^{(\pm)}$ by (4.32). We will show that the result coincides with the conditions obtained from the currents in subsection 4.2, and hence that the above action is indeed the correct one.

The general field variation of (4.39) is

$$\delta S = i\int d\tau \Big[(\delta\psi_+^\mu\psi_+^\nu - \delta\psi_-^\mu\psi_-^\nu)g_{\mu\nu} + \\ + \delta X^\mu(i\partial_{+\!\!+} X^\nu E_{\nu\mu} - i\partial_= X^\nu E_{\mu\nu} + \Gamma^-_{\nu\mu\rho}\psi_-^\nu\psi_-^\rho - \Gamma^+_{\nu\mu\rho}\psi_+^\nu\psi_+^\rho)\Big]. \tag{4.41}$$

The variation of (4.39) under the manifest supersymmetry (4.9) is

$$\delta_{s1} S = \epsilon_1^- \int d\tau \Big[\partial_{\mp\!\mp} X^\mu \psi_-^\nu E_{\mu\nu} - \eta_1 \psi_+^\mu \partial_= X^\nu E_{\mu\nu} + \eta_1 \partial_{\mp\!\mp} X^\mu \psi_+^\nu B_{\mu\nu} + \partial_= X^\mu \psi_-^\nu B_{\mu\nu} - \\ - \frac{i}{3} \eta_1 H_{\mu\nu\rho} \psi_+^\rho \psi_+^\mu \psi_+^\nu - \frac{i}{3} H_{\mu\nu\rho} \psi_-^\rho \psi_-^\mu \psi_-^\nu \Big] . \tag{4.42}$$

The variation of (4.39) under the nonmanifest supersymmetry (4.10) is

$$\delta_{s2} S = \epsilon_2^- \int d\tau \Big[\eta_2 \partial_= X^\mu \psi_+^\rho J^\nu_{+\rho} E_{\nu\mu} - \partial_{\mp\!\mp} X^\mu \psi_-^\rho J^\nu_{-\rho} E_{\mu\nu} - \eta_2 \partial_{\mp\!\mp} X^\mu \psi_+^\rho J^\nu_{+\rho} B_{\mu\nu} - \\ - \partial_= X^\mu \psi_-^\rho J^\nu_{-\rho} B_{\mu\nu} + i J^\lambda_{-\mu} H_{\lambda\nu\rho} \psi_-^\mu \psi_-^\nu \psi_-^\rho + i\eta_2 J^\lambda_{+\mu} H_{\lambda\nu\rho} \psi_+^\mu \psi_+^\nu \psi_+^\rho \Big] . \tag{4.43}$$

Starting from the fermionic ansatz (3.6) we have to find boundary conditions that set the three variations (4.41), (4.42) and (4.43) to zero. Using the results of [10] we only need to deal with the last variation, (4.43). After straightforward calculations we find that the $N = 1$ conditions have to be supplemented by the single condition (4.34). Thus the action (4.39) reproduces the boundary conditions derived from the currents.

For commuting complex structures there is an off-shell $N = 2$ represenation of the model in terms of chiral and twisted chiral superfields. The action is given by (4.8) and there is a gauge symmetry for the generalized Kähler potential

$$K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}) \quad \rightarrow \quad K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}) + h(\Phi) + \bar{h}(\bar{\Phi}) + g(\Lambda) + \bar{g}(\bar{\Lambda}) . \tag{4.44}$$

In complete analogy with the Kähler case this gauge symmetry is broken by the boundary terms.

4.4 Geometry

In the previous subsection we analysed the formal aspects of the $N = 2$ superconformal boundary conditions for the bihermitean case and found the most general local $N = 2$ superconformal boundary conditions. Following the Kähler case we would like to find a geometrical interpretation of these boundary conditions in terms of special types of submanifolds. Unfortunately now we cannot use the results from symplecic geometry which we used in the analysis of the Kähler case. For a $N = 2$ sigma models with torsion the target manifold is a bihermitean manifold with the additional property (4.6). As far as we are aware there is no clear geometrical interpretation of the non-linear condition (4.6) which relates two complex structures. Therefore we cannot give a full interpretation of the most general solution of the model in simple geometrical terms. Nevertheless we present some partial results on the geometry of these boundary conditions.

In contrast to the Kähler case the corresponding submanifold cannot be invariant (anti-invariant) with respecet to one of the complex structures. To illustrate this point, let us assume that a submanifold is J_--invaraint (i.e., $[R, J_-] = 0$). We then have the following

$$(\eta_1\eta_2) R J_+ = J_- R = R J_- , \quad \rightarrow \quad R(J_- - (\eta_1\eta_2) J_+) = 0 . \tag{4.45}$$

Since by the definition R is nondegenerate we run into the contradiction that $J_+ = \pm J_-$. Thus R cannot carry purely holomorphic (antiholomorphic) indices with respect to one of the complex structures (J_+, e.g).

The next property which is very special for the present model is the relation between the torsion, H and the boundary condition, R. As we have discussed there is a non trivial relation (4.5) between H and $J_\pm$ which follows from the inegrability of $J_\pm$ and the covariant constancy of $J_\pm$ with respect to the affine connections with torsion. In the complex coordinates for J_+ (J_-) the condition (4.5) have a simple form: $H_{ijk} = 0$ and $H_{\bar{i}\bar{j}\bar{k}} = 0$. Combining the property (4.14) with (4.5) we obtain a condition which involves R

$$\begin{cases} H^R_{\delta\nu\lambda} = J^\sigma_{+\delta} J^\rho_{+\nu} H^R_{\sigma\rho\lambda} + J^\sigma_{+\lambda} J^\rho_{+\delta} H^R_{\sigma\rho\nu} + J^\sigma_{+\nu} J^\rho_{+\lambda} H^R_{\sigma\rho\delta} \\ H_R^{\delta\nu\lambda} = J^\delta_{-\sigma} J^\nu_{-\rho} H_R^{\sigma\rho\lambda} + J^\lambda_{-\sigma} J^\delta_{-\rho} H_R^{\sigma\rho\nu} + J^\nu_{-\sigma} J^\lambda_{-\rho} H_R^{\sigma\rho\delta} \end{cases} \tag{4.46}$$

with the following notation

$$H^R_{\delta\nu\lambda} \equiv R^\mu_{\ \delta} R^\rho_{\ \nu} R^\sigma_{\ \lambda} H_{\mu\rho\sigma}\,, \qquad H_R^{\delta\nu\lambda} \equiv R^\delta_{\ \mu} R^\nu_{\ \rho} R^\lambda_{\ \sigma} H^{\mu\rho\sigma}\,. \tag{4.47}$$

In canonical coordinates for J_+ the property (4.46) has the relatively simple form

$$R^\mu_{\ i} R^\rho_{\ j} R^\sigma_{\ k} H_{\mu\rho\sigma} = 0\,, \qquad R^\mu_{\ \bar{i}} R^\rho_{\ \bar{j}} R^\sigma_{\ \bar{k}} H_{\mu\rho\sigma} = 0 \tag{4.48}$$

where i, j, k ($\bar{i}, \bar{j}, \bar{k}$) are holomorphic indices and $\mu = (i, \bar{i})$. Since R can neither commute nor anticommute with J_+ the relation (4.48) is a nontrivial restriction on possible R's.

Another common property of these branes is that

$$Q(J_+ - (\eta_1\eta_2)J_-)Q = 0 \tag{4.49}$$

which follows from (4.14) and the property $RQ = QR = -Q$. In geometrical terms the property (4.49) says that a linear combination of two forms $J_{\pm\mu\nu}$ is zero when restriced to the normal space

$$(J_{+\mu\nu} - (\eta_1\eta_2)J_{-\mu\nu})|_{\mathcal{N}_X(D)} = 0\,. \tag{4.50}$$

Next we look at the special subclass of the boundary conditions with the property that $R^2 = I$. In this case it is convenient to introduce two (1,1) tensors

$$L^\mu_{\ \nu} \equiv J^\mu_{+\nu} + J^\mu_{-\nu}\,, \qquad M^\mu_{\ \nu} \equiv J^\mu_{+\nu} - J^\mu_{-\nu} \tag{4.51}$$

which may be degenerate in a generic situation. Using these tensors we rewrite the condition (4.14) as

$$R^\mu_{\ \rho} L^\rho_{\ \sigma} R^\sigma_{\ \nu} = (\eta_1\eta_2) L^\mu_{\ \nu}\,, \qquad R^\mu_{\ \rho} M^\rho_{\ \sigma} R^\sigma_{\ \nu} = -(\eta_1\eta_2) M^\mu_{\ \nu}\,. \tag{4.52}$$

Alternatively since the metric is hermitean with respect to both complex structures there exist two corresponding two forms $M_{\mu\nu}$ and $L_{\mu\nu}$. In terms of these antisymmetric tensors the condition (4.52) is rewritten as

$$R^\mu_{\ \rho} L_{\mu\nu} R^\nu_{\ \sigma} = (\eta_1\eta_2) L_{\rho\sigma}\,, \qquad R^\mu_{\ \rho} M_{\mu\nu} R^\nu_{\ \sigma} = -(\eta_1\eta_2) M_{\rho\sigma}\,. \tag{4.53}$$

We first consider the case when $(\eta_1\eta_2) = 1$. The relation (4.52) takes the form

$$[R, L] = 0\,, \qquad \{R, M\} = 0\,. \tag{4.54}$$

In full analogy with the analysis from the section 3 we find that the resulting submanifold D is an invariant submanifold with respect to L and anti-invariant with respect to M, i.e.

$$L\mathcal{T}_X(D) \subset \mathcal{T}_X(D)\,, \qquad L\mathcal{N}_X(D) \subset \mathcal{N}_X(D)\,, \tag{4.55}$$

$$M\mathcal{T}_X(D) \subset \mathcal{N}_X(D)\,, \qquad M\mathcal{N}_X(D) \subset \mathcal{T}_X(D). \tag{4.56}$$

For the case $(\eta_1\eta_2) = -1$ the role of L and M is interchanged. Thus the resulting manifold is L-anti-invariant and M-invariant.

Alternatively we may construct other (1,1)-tensors, for example the commutator and anticommutator of the complex structures. In this case the condition (4.14) together with $R^2 = I$ implies

$$[J_-, J_+]R = -R[J_-, J_+]\,, \qquad \{J_-, J_+\}R = R\{J_-, J_+\}\,. \tag{4.57}$$

Thus the submanifold is anti-invariant with respect to $[J_-, J_+]$ and invariant with respect to $\{J_+, J_-\}$ (unless one of them are zero).

If L and M are non degenerate, then $\ker[J_+, J_-] = \emptyset$, where

$$\ker[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)\,. \tag{4.58}$$

In this case the backgound manifold has to be $4k$ dimensional [17]. It is easy to see that when the antisymmetric tensors $L_{\mu\nu}$ and $M_{\mu\nu}$ are non degenerate the conditions (4.53) do not have a solution such that $R^2 = I$. In the next subsection we will analyse the opposite case when $[J_+, J_-] = 0$ and, as we will see, there are many solutions which satisfy $R^2 = I$. Therefore we conclude that $\ker[J_+, J_-]$ "controls" the solutions with $R^2 = I$ (i.e., without B-field).

It is difficult to describe the most general case exhaustively. Using L and M, the condition (4.14) for the case $(\eta_1\eta_2) = 1$ can be rewritten as

$$[R, L] = T\,, \qquad \{R, M\} = T \tag{4.59}$$

where T is an auxilary object. For the case $(\eta_1\eta_2) = -1$ L and M would be interchanged in (4.59). The fact that $J_\pm$ are complex structures implies that

$$\{L, M\} = 0\,, \qquad L^2 + M^2 = -2I\,. \tag{4.60}$$

If we consider the space-filling brane then R is a globally defined $(1,1)$ tensor satisfying the algebra (4.59) and (4.60). The existence of this algebra on the target manifold should have non trivial consequences. However, we are not familiar with this type of the structures in the mathematical literature.

4.4.1 Locally product manifolds

In this subsection we consider the special case of commuting comlex structures,

$$[J_+, J_-]^\mu{}_\nu \equiv J^\mu_{+\rho} J^\rho_{-\nu} - J^\mu_{-\rho} J^\rho_{+\nu} = 0\,. \tag{4.61}$$

Then the tensor

$$\Pi^\mu{}_\nu \equiv J^\mu_{+\rho} J^\rho_{-\nu}\,, \tag{4.62}$$

satisfies

$$\Pi^\mu{}_\rho \Pi^\rho{}_\nu = \delta^\mu{}_\nu\,. \tag{4.63}$$

A tensor satsifying above requirement is known as an almost product structure [12]. As a result of the inegrability of $J_\pm$ the almost product structure Π is also integrable. An integrable $(1,1)$ tensor with the property (4.63) gives rise to a local product structure. Using the property (4.6) we see that the submanifolds projected out by

$$\Pi_\pm = \frac{1}{2}(I \pm \Pi) \tag{4.64}$$

are Kähler. Thus the geometry locally looks like a product of two Kähler manifolds, $\mathcal{M}_1 \times \mathcal{M}_2$. However the geometry is not a locally decomposable riemannian manifold, i.e. the metric on $\mathcal{M}_1$ depends on the coordinates of $\mathcal{M}_2$ and vice versa. An example where this situation is realized is the group manifold, $\mathrm{SU}(2) \times \mathrm{U}(1)$. However, in this example the local product structure Π is not the obvious one.

If we consider solutions with the property $R^2 = I$ then, as result of (4.14), we get

$$[\Pi, R] = 0\,. \tag{4.65}$$

Thus in this case the submanifold corresponding to a D-brane is Π-invariant, i.e.

$$\Pi \mathcal{T}_X(D) \subset \mathcal{T}_X(D)\,, \qquad \Pi \mathcal{N}_X(D) \subset \mathcal{N}_X(D)\,. \tag{4.66}$$

Therefore Π induces a local product structure on the brane, D. R is decomposable in the following form

$$R = \Pi_+ R \Pi_+ + \Pi_- R \Pi_- \equiv R^+ + R^-\,. \tag{4.67}$$

Using the above decomposition the problem can be completly separated into two independent problems for the Kähler geometry

$$\begin{cases} R^+ J_+ + (\eta_1\eta_2) R^+ J_+ = 0 \\ R^- J_+ - (\eta_1\eta_2) R^- J_+ = 0 \end{cases} \tag{4.68}$$

where the first equation is understood on $\mathcal{M}_1$ and the second on $\mathcal{M}_2$. Thus for the the case $(\eta_1\eta_2) = 1$ we have

$$\{R^+, J_+\} = 0\,, \qquad [R^-, J_+] = 0 \tag{4.69}$$

and the resulting brane is a local product of a lagrangian submanifold of $\mathcal{M}_1$ and a Kähler submanifold of $\mathcal{M}_2$ with respect to J_+. When $(\eta_1\eta_2) = -1$ the situation is interchanged, i.e. the resulting brane is local product of a lagrangian submanifold of $\mathcal{M}_2$ and a Kähler submanifold of $\mathcal{M}_1$ with respect to J_+. Thus unlike the case with $\ker[J_+, J_-] = \emptyset$ there are a lot of solutions with the property $R^2 = I$.

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Next if we look for more general solutions with $R^2 \neq I$ then we still construct them in the same fashion assuming the property (4.65). However there are solutions whith $R^2 \neq I$ which do not obey the condition (4.65) and thus they cannot be thought of as local products of Kähler branes. In the general situation we solve the problem explicitly in special coordinates. Coordinates exist in which Π and $J_{\pm}$ take block diagonal forms

$$\Pi^{\mu}{}_{\nu} = \begin{pmatrix} -\delta^{n}{}_{m} & 0 & 0 & 0 \\ 0 & -\delta^{\bar{n}}{}_{\bar{m}} & 0 & 0 \\ 0 & 0 & \delta^{i}{}_{j} & 0 \\ 0 & 0 & 0 & \delta^{\bar{i}}{}_{\bar{j}} \end{pmatrix}, \tag{4.70}$$

$$J^{\mu}_{-\nu} = \begin{pmatrix} i\delta^{n}{}_{m} & 0 & 0 & 0 \\ 0 & -i\delta^{\bar{n}}{}_{\bar{m}} & 0 & 0 \\ 0 & 0 & i\delta^{i}{}_{j} & 0 \\ 0 & 0 & 0 & -i\delta^{\bar{i}}{}_{\bar{j}} \end{pmatrix}, \qquad J^{\mu}_{+\mu} = \begin{pmatrix} -i\delta^{n}{}_{m} & 0 & 0 & 0 \\ 0 & i\delta^{\bar{n}}{}_{\bar{m}} & 0 & 0 \\ 0 & 0 & i\delta^{i}{}_{j} & 0 \\ 0 & 0 & 0 & -i\delta^{\bar{i}}{}_{\bar{j}} \end{pmatrix}. \tag{4.71}$$

Thus the condition (4.14) with $(\eta_1\eta_2) = 1$ can be solved for R as follows

$$R^{\mu}{}_{\nu} = \begin{pmatrix} 0 & R^{n}{}_{\bar{m}} & R^{n}{}_{j} & 0 \\ R^{\bar{m}}{}_{n} & 0 & 0 & R^{\bar{m}}{}_{\bar{j}} \\ 0 & R^{i}{}_{\bar{m}} & R^{i}{}_{j} & 0 \\ R^{\bar{i}}{}_{n} & 0 & 0 & R^{\bar{i}}{}_{\bar{j}} \end{pmatrix}. \tag{4.72}$$

We see that R does not have a block diagonal form and in the general situation the property (4.65) is not true. There are other condition which restricts R further. For example, the condition which involve the metric $R^t g R = g$. The metric g is bihermitean with respect to both complex structure and has the special form

$$g_{\mu\nu} = \begin{pmatrix} 0 & g_{n\bar{m}} & 0 & 0 \\ g_{n\bar{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{i\bar{j}} \\ 0 & 0 & g_{i\bar{j}} & 0 \end{pmatrix}. \tag{4.73}$$

However the condition involving the metric does not restrict R to be block diagonal in the Π-coordinates. Other properties of the branes would depend on the specific properties of the model. A similar analysis can be done for the case $(\eta_1\eta_2) = -1$.

5. $N = 2$ Landau-Ginzburg models

In this section we consider the special subclass of boundary conditions for the massive generalization of the $N = 2$ sigma models, so called $N = 2$ Landau-Ginzburg models.

The $N = 1$ Landau-Ginzburg model is given by the bulk action

$$S = \int d^2\sigma \, d^2\theta \, [D_+\Phi^\mu D_-\Phi^\nu (g_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)) + W(\Phi)] \,. \tag{5.1}$$

Generically the classical model is not conformally invariant because of the presence of the potential W. However it shares some interesting properties with the sigma model.

Boundary conditions for the $N = 1$ Landau-Ginzburg model in a trivial background are discussed in [6]. In particular a boundary potential is found to be necessary for supersymmetry. In this section we discuss the $N = 2$ version of the Landau-Ginzburg model. The bulk action is manifestly $N = 1$ supersymmetric. Depending on whether the field strength for B is zero or nonzero, the nonmanifest supersymmetry has either the form (3.2) or the form (4.2). Therefore the last term in (5.1) should be supersymmetric by itself.

We first analyse the Kähler case. The variation of the last term in (5.1) with respect to the nonmanifest supersymmetry (3.2) is

$$\delta \int d^2\sigma\, d^2\theta\; W(\Phi) = \int d^2\sigma\, d^2\theta\; W_{,\mu} J^{\mu}_{\ \nu} \epsilon_2^{\alpha} D_{\alpha} \Phi^{\nu}\,. \tag{5.2}$$

We require the integrand to be a total derivative and thus find

$$J^{\mu}_{\ \nu} \partial_{\mu} W = \partial_{\nu} \tilde{W} \tag{5.3}$$

where $\tilde{W}$ is an arbitrary function. This condition is a higher dimensional analog of the Cauchy-Riemann equations and thus W and $\tilde{W}$ can be thought of as the real and imaginary parts of a holomorphic function, $\mathcal{W} = W + i\tilde{W}$. Equation (5.3) implies the following integrability conditions

$$\partial_{[\rho}(J^{\mu}_{\ \nu]} \partial_{\mu} W) = 0\,. \tag{5.4}$$

In two dimensions this condition is the requirement that the real part of holomorphic function should be harmonic. The bosonic potential can be written either in terms of W (the real part) or in terms of $\tilde{W}$ (the imaginary part)

$$V(X) = \frac{1}{4} \partial_{\mu} W \partial_{\nu} W g^{\mu\nu} = \frac{1}{4} \partial_{\mu} \tilde{W} \partial_{\nu} \tilde{W} g^{\mu\nu}\,. \tag{5.5}$$

When the taget space manifold is Kähler and W satisfies (5.4) the action can be rewritten in a manifestly $N = 2$ supersymmetric form

$$S = \int d^2\sigma\, d^2\theta\, d^2\bar{\theta}\; K(\Phi, \bar{\Phi}) + \int d^2\sigma\, d^2\theta\; \mathcal{W}(\Phi) + \int d^2\sigma\, d^2\bar{\theta}\; \bar{\mathcal{W}}(\bar{\Phi}) \tag{5.6}$$

where $\mathcal{W}$ is the holomorphic prepotential defined above.

If $dB \neq 0$, the taget manifold should be bihermitean and the nonmanifest supersymmetry is given by (4.2). As in the Kähler case we require the potential term to be supersymmetric by itself. Following the previous line of argument, we take the supersymmetry variation of the potential term to be a total derivative and as result we get two conditions

$$J^{\mu}_{+\nu} \partial_{\mu} W = \partial_{\nu} \tilde{W}_{+}\,, \qquad J^{\mu}_{-\nu} \partial_{\mu} W = \partial_{\nu} \tilde{W}_{-} \tag{5.7}$$

where $\tilde{W}_{+}$ and $\tilde{W}_{-}$ are arbitrary functions. Equations (5.7) imply integrability conditions for W;

$$\partial_{[\rho}(J^{\mu}_{+\nu]} \partial_{\mu} W) = 0\,, \qquad \partial_{[\rho}(J^{\mu}_{-\nu]} \partial_{\mu} W) = 0\,. \tag{5.8}$$

Therefore we may form two complex functions, $\mathcal{W}_{+} = W + i\tilde{W}_{+}$ and $\mathcal{W}_{-} = W + i\tilde{W}_{-}$ which are both holomorphic but with respect to different complex structures. If the two complex structures commute we may use a manifest $N = 2$ formalism with chiral (antichiral) and twisted chiral (antichiral) supersfields.

In what follows we will consider a special class of boundary conditions for $N = 2$ Landau-Ginzburg models which admits a nice geometrical interpretation. For the Kähler case the $N = 2$ boundary conditions were studied in [5] and [20] (for earlier study see [21]). We will reproduce some of the results of [5] (but in a different fashion) and we will comment on them. We do not know of a study of $N = 2$ boundary conditions for the bihermitean Landau-Ginzburg model.

5.1 Kähler Landau-Ginzburg model

We would like to look at a special subclass of $N = 2$ supersymmetric boundary conditions for the Kähler Landau-Ginzburg model. The on-shell supersymmetry transformations have an extra term related to the potential compared with the sigma model. In components the manifest on-shell supersymmetry transformation are

$$\begin{cases} \delta_1 X^\mu = -(\epsilon_1^+ \psi_+^\mu + \epsilon_1^- \psi_-^\mu) \\ \delta_1 \psi_+^\mu = -i\epsilon_1^+ \partial_{\#} X^\mu + \epsilon_1^- \Gamma^\mu{}_{\nu\rho} \psi_-^\rho \psi_+^\nu + \frac{1}{2}\epsilon_1^- g^{\mu\nu} W_{,\nu} \\ \delta_1 \psi_-^\mu = -i\epsilon_1^- \partial_{=} X^\mu - \epsilon_1^+ \Gamma^\mu{}_{\nu\rho} \psi_-^\rho \psi_+^\nu - \frac{1}{2}\epsilon_1^+ g^{\mu\nu} W_{,\nu} \end{cases} \tag{5.9}$$

and the nonmanifest ones are

$$\begin{cases} \delta_2 X^\mu = (\epsilon_2^+ \psi_+^\nu + \epsilon_2^- \psi_-^\nu) J^\mu_\nu \\ \delta_2 \psi_+^\mu = -i\epsilon_2^+ \partial_{\#} X^\nu J^\mu_\nu - \epsilon_2^- J^\mu_\sigma \Gamma^\sigma{}_{\nu\rho} \psi_-^\rho \psi_+^\nu - \epsilon_2^+ J^\mu_{\nu,\rho} \psi_+^\rho \psi_+^\nu - \epsilon_2^- J^\mu_{\nu,\rho} \psi_+^\rho \psi_-^\nu - \\ \qquad - \frac{1}{2}\epsilon_2^- J^\mu{}_\nu g^{\nu\rho} W_{,\rho} \\ \delta_2 \psi_-^\mu = -i\epsilon_2^- \partial_{=} X^\nu J^\mu_\nu + \epsilon_2^+ J^\mu_\sigma \Gamma^\sigma{}_{\nu\rho} \psi_-^\rho \psi_+^\nu - \epsilon_2^+ J^\mu_{\nu,\rho} \psi_-^\rho \psi_+^\nu - \epsilon_2^- J^\mu_{\nu,\rho} \psi_-^\rho \psi_-^\nu + \\ \qquad + \frac{1}{2}\epsilon_2^+ J^\mu{}_\nu g^{\nu\rho} W_{,\rho} \end{cases} \tag{5.10}$$

As before start from the fermion ansatz

$$\psi_-^\mu = \eta_1 R^\mu{}_\nu \psi_+^\nu \, . \tag{5.11}$$

However unlike the classical sigma model for Landau-Ginzburg model we cannot argue that this ansatz is the unique local ansatz for the fermions since there is a dimensionful coupling. In fact, the characteristic property of (5.11) is that it is a local ansatz which does not contain a dimensionful parameter. We will study only this type of boundary conditions.

The first supersymmetry transformation (5.9) applied to (5.11) yields

$$\partial_{=} X^\mu - R^\mu{}_\nu \partial_{\#} X^\nu + 2i P^\rho_\gamma \nabla_\rho R^\mu{}_\nu \psi_+^\gamma \psi_+^\nu - i\eta_1 P^\mu_\nu g^{\nu\rho} W_{,\rho} = 0 \, . \tag{5.12}$$

This is then the bosonic $N = 1$ boundary condition. An important point is that the condition (5.12) still has a geometrical inerpretation in terms of submanifolds since the last term in (5.12) does not affect the derivation of the inegrability conditions (2.11).

From the second supersymmetry variation of (5.11) we get

$$\partial_{=} X^\mu + (\eta_1\eta_2) J^\mu_\sigma R^\sigma{}_\nu J^\nu{}_\gamma \partial_{\#} X^\gamma + i\Big[(\eta_1\eta_2) J^\mu_\sigma \nabla_\rho R^\sigma{}_\nu J^\rho{}_\gamma + \\ + J^\mu_\sigma \nabla_\rho R^\sigma{}_\nu J^\rho{}_\lambda R^\lambda{}_\gamma\Big] \psi_+^\gamma \psi_+^\nu + i\eta_2 P^\mu_\nu g^{\nu\rho} W_{,\rho} = 0 \, . \tag{5.13}$$

The boundary conditions (5.12) and (5.13) should be equivalent. Previously we have discussed the corresponding boundary conditions for the Kähler sigma model. Now for the Landau-Gizburg model the only difference is the last term in (5.12) and (5.13). Therefore all previous results from the sigma model analysis apply here as well, including the geometrical interpretation. The only new ingredient is that $N = 2$ supersymmetry requires the following property of the potential on the boundary

$$(\eta_1 + \eta_2) P^{\mu}_{\nu} g^{\nu\rho} W_{,\rho} = 0 \,. \tag{5.14}$$

When $\eta_1 = \eta_2$ (i.e., $\eta_1\eta_2 = 1$) this does not vanish identically. This corresponds to the B-type condition. Using the B-type property that $PJ = JP$ we see that the holomorphic prepotential $\mathcal{W}$ is constant along the B-type submanifold since

$$P^{\mu}_{\nu} g^{\nu\rho} W_{,\rho} = 0 \,, \qquad P^{\mu}_{\nu} g^{\nu\rho} \tilde{W}_{,\rho} = 0 \,. \tag{5.15}$$

This result completly agrees with [5].

However for the A-type boundary condition the requirement (5.14) is automatically satisfied. Thus the last term in the bosonic boundary conditions (5.12) is allowed for the A-type supersymmetry. To understand the meaning of this term we ask about the dynamical nature of these boundary conditions. Using the properties (2.23) we rewrite the bosonic boundary conditions in an equivalent form which may be derived from the following action

$$S = S_{bulk} + \int d\tau V_b(X) \,, \qquad (V_{b,\mu} + i\eta_1 W_{,\mu})|_{\mathcal{T}_X(D)} = 0 \tag{5.16}$$

where S_{bulk} is the sigma model action (3.61). Thus (as in the $N = 1$ case [6]) there is a boundary potential V_b which is given by the real part (or the imaginary part, depending on the conventions) of the bulk prepotential $\mathcal{W}$. We stress that this result is valid for all A-type branes (i.e., even with a B-field).

There is interesting subclass of the above boundary conditions. We can require that the bosonic boundary condition does not contain a dimensionful parameter. Thus the boundary potential should be constant along the brane, i.e.

$$\pi^{\nu}{}_{\mu} W_{,\nu} = 0 \,. \tag{5.17}$$

We conclude that the real (or imaginary) part of the prepotential is constant along brane,

$$Re(\mathcal{W}_{,\mu})|_{\mathcal{T}_X(D)} = 0 \,. \tag{5.18}$$

This result agrees completly with [5] (modulo the conventions related to the supersymmetry transformations). However, the derivation persented here is somewhat different.

We stress that we do not discuss the general problem of the introduction of a boundary potential for the sigma model. In the absence of the bulk potential the introduction of a supersymmetric boundary potential would require non-locality (i.e., the auxilary fields) on the boundary, see for example [22] and [23]. In the our case this non-locality is avoided by the presence of the bulk potential in the on-shell supersymmetry transformations.

5.2 Bihermitean Landau-Ginzburg model

We can repeat a similar analysis for the boundary conditions of the bihermitean Landau-Ginzburg model. However we face the same problem as for the bihermitean sigma model, namely a lack of geometrical understanding of the whole background and consequently of the allowed branes on it. Nevertheless below we go through the formal analysis and for the case of commuting structure we interpret the result.

The manifest on-shell supersymmetry transformations are

$$\begin{cases} \delta_1 X^\mu = -\left(\epsilon_1^+ \psi_+^\mu + \epsilon_1^- \psi_-^\mu\right) \\ \delta_1 \psi_+^\mu = -i\epsilon_1^+ \partial_{+\!\!+} X^\mu + \epsilon_1^- \Gamma^{-\mu}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu + \frac{1}{2}\epsilon_1^- g^{\mu\nu} W_{,\nu} \\ \delta_1 \psi_-^\mu = -i\epsilon_1^- \partial_{=} X^\mu - \epsilon_1^+ \Gamma^{-\mu}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu - \frac{1}{2}\epsilon_1^+ g^{\mu\nu} W_{,\nu} \end{cases} \tag{5.19}$$

and the nonmanifest supersymmetry transformations (4.2) are

$$\begin{cases} \delta_2 X^\mu = \epsilon_2^+ \psi_+^\nu J^\mu_{+\nu} + \epsilon_2^- \psi_-^\nu J^\mu_{-\nu} \\ \delta_2 \psi_+^\mu = -i\epsilon_2^+ \partial_{+\!\!+} X^\nu J^\mu_{+\nu} - \epsilon_2^- J^\mu_{-\sigma} \Gamma^{-\sigma}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu + \epsilon_2^+ J^\mu_{+\nu,\rho} \psi_+^\nu \psi_+^\rho + \\ \qquad + \epsilon_2^- J^\mu_{-\nu,\rho} \psi_-^\nu \psi_+^\rho - \frac{1}{2}\epsilon_2^- J^\mu_{-\sigma} g^{\sigma\lambda} W_{,\lambda} \\ \delta_2 \psi_-^\mu = -i\epsilon_2^- \partial_{=} X^\nu J^\mu_{-\nu} + \epsilon_2^+ J^\mu_{+\sigma} \Gamma^{-\sigma}_{\ \nu\rho} \psi_-^\rho \psi_+^\nu + \epsilon_2^+ J^\mu_{+\nu,\rho} \psi_+^\nu \psi_-^\rho + \\ \qquad + \epsilon_2^- J^\mu_{-\nu,\rho} \psi_-^\nu \psi_-^\rho + \frac{1}{2}\epsilon_2^+ J^\mu_{+\sigma} g^{\sigma\lambda} W_{,\lambda}\,. \end{cases} \tag{5.20}$$

With the analysis in the previous subsection in mind we derive the special bosonic boundary conditions. Starting from the fermionic ansatz (5.11) and acting with the first supersymmetry we find

$$\partial_= X^\mu - R^\mu_{\ \nu} \partial_{+\!\!+} X^\nu + 2i(P^\sigma_{\ \gamma} \nabla_\sigma R^\mu_{\ \nu} + P^\mu_{\ \rho} g^{\rho\delta} H_{\delta\sigma\gamma} R^\sigma_{\ \nu}) \psi_+^\gamma \psi_+^\nu - i\eta_1 P^\mu_{\ \nu} g^{\nu\rho} W_{,\rho} = 0 \tag{5.21}$$

where $\epsilon_1^+ = \eta_1 \epsilon_1^-$. The second supersymmetry gives

$$\partial_= X^\mu + (\eta_1\eta_2) J^\mu_{-\lambda} R^\lambda_{\ \sigma} J^\sigma_{+\nu} \partial_{+\!\!+} X^\nu + \ldots - \frac{i}{2}\left(\eta_2 J^\mu_{-\lambda} J^\lambda_{+\sigma} + \eta_1 J^\mu_{-\gamma} R^\gamma_{\ \lambda} J^\lambda_{-\sigma}\right) g^{\sigma\rho} W_{,\rho} = 0 \tag{5.22}$$

where the dots stand for the same two fermion term as in equation (4.12). The boundary conditions (5.21) and (5.22) should be equivalent. Previously we have analyzed the X-part and two-fermion term for the corresponding $N = 2$ sigma model. This analysis extends to the present case of the Landau-Ginzburg model. The only new ingradient is that the $N = 2$ supersymmetry requires the following property of the potential on the boundary

$$\left(\delta^\mu_{\ \sigma} + R^\mu_{\ \sigma} - (\eta_1\eta_2) J^\mu_{-\lambda} J^\lambda_{+\sigma} - (\eta_1\eta_2) R^\mu_{\ \gamma} J^\gamma_{+\lambda} J^\lambda_{-\sigma}\right) g^{\sigma\rho} W_{,\rho} = 0 \tag{5.23}$$

This condition can be understood as a requirement for a brane, R, with a given potential, W. Following the previous subsection, on the boundary W would be interpreted as a boundary potential. If we restrict ourselves to the case $R^2 = I$ the condition (5.23) can be rewritten in the form

$$(J^\mu_{-\gamma} + (\eta_1\eta_2) J^\mu_{+\gamma}) P^\gamma_{\ \lambda} g^{\lambda\rho} W_{,\rho} = 0\,, \tag{5.24}$$

where we have used the property (4.14). We conclude that for branes with $R^2 = I$ the structure of $\ker[J_+, J_-]$ (see equation (4.58)) defines the possible restriction on the potential.

For the case of commuting complex structures the interpretation of the above conditions is clear. The condition (5.23) may be rewritten in the form

$$(P^{\mu}_{\ \sigma} - (\eta_1\eta_2)P^{\mu}_{\ \lambda}\Pi^{\lambda}_{\ \sigma})g^{\sigma\rho}W_{,\rho} = 0 \tag{5.25}$$

where Π is the product structure from subsection 4.4.1. For the case of $(\eta_1\eta_2) = 1$ boundary conditions we get

$$P^{\mu}_{\ \lambda}\Pi^{\lambda}_{-\sigma}g^{\sigma\rho}W_{,\rho} = 0 \tag{5.26}$$

and for the case $(\eta_1\eta_2) = -1$

$$P^{\mu}_{\ \lambda}\Pi^{\lambda}_{+\sigma}g^{\sigma\rho}W_{,\rho} = 0\,. \tag{5.27}$$

For the composite branes (i.e., $[\Pi, R] = 0$) the conditions (5.26) and (5.27) say that one has to have a constant W along the B-part of the brane. Thus it agrees completely with the discussion of the Kähler case. However the relations (5.26) and (5.27) remain true even for the case of non-decomposeable branes.

In analogy with the Kähler case we can consider the situation when the bosonic boundary condition does not contain a dimensionful parameter.

6. Discussion

We have presented a detailed analysis of the local superconformal boundary conditions for $N = 2$ sigma models. Our analysis represents the most general case in the sense described in the introduction. Namely, there are various different conditions for boundary symmetries that we use, closure of the algebra, gluing of the currents etc., and our results ensure that they are all satisfied. Our starting point was the general local classical condition on the fermions, (2.3).

For the Kähler case we reproduce results in a systematic fashion and present an analysis of non-lagrangian A-branes. Recently non-lagrangian A-branes have also been discussed in [11].

We stress that in the most general local boundary conditions, the bosonic condition has a two fermion term unless the brane is totally geodesic. Since there appears to be some confusion about these issues in the literuture we thought it instructive to go through the details of the general derivation of the boundary conditions for the Kähler sigma model.

Another type of $N = 2$ sigma model has a bihermitean target space with extra properties. This models necessarily involves torsion (a non zero field strength for the NS-NS two form). For bihermitean sigma models we give the full analysis of the local superconformal boundary conditions. Also we discuss the geometrical interpretation of these boundary conditions.

As a natural supplement to the sigma model discussion we consider their massive generalization, the Landau-Ginzburg model. We describe a special subclass of boundary conditions of the Landau-Ginzburg model for which we have a nice geometrical interpretation as submanifolds.

The main motiovation for our investigation comes from string theory. However since our analysis lies entirely in the realm of classical field theory and we have tried to maintain a certain level of mathematical rigour, the present results may be useful for mathematical physics as well. Further, from a string theory point of view the sigma model arise as gauge fixed version of the open string action. Correspondingly a complete analysis should also take the BRST symmetry for the gauge fixed worldsheet diffeomorphisms into account. For certain boundary conditions and trivial background this is done for $N = 1$ in [6], but there are still open problems in this context.

One interesting aspects of our analysis is that the consistent open string sigma models (i.e., the sigma model with boundaries) may require a new geometry on the target space. If we consider freely moving strings (i.e., the space-filling brane) we have to introduce a globally defined $(1,1)$ tensor field $R^{\mu}_{\ \nu}$ which encodes the boundary conditions for a sigma model at hand. Requiring certain symmetries of the model to be preserved in the presence of the boundary amounts to conditions on $R^{\mu}_{\ \nu}$ and therefore possible to new geometry. For example, the A-type Kähler sigma model (i.e., the sigma model with the A-type boundary conditions) would require an extra complex structure, $\tilde{J}$ with the property (3.36). In the bihermitean case freely moving strings lead to the structure (4.59) and (4.60) which we do not know how to interpret at the moment. We think it worthwhile to pursue this topic further.

Many open problems remain in the subject. One of them is the search for a better geometrical understanding of the bihermitean geometry which admits $N = 2$ sigma models with torsion. Such an understanding will shed light on the branes which this geometry admits. Another topic would be a detailed analysis (in the spirit of the present work) of the semiclassical branes on Calabi-Yau manifolds. Although there are many things which are known about these (see, e.g., [24] and [25]), we feel that a rigorous semiclassical analysis would be useful. We hope to come back to these problems elsewhere.

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A. (1,1) supersymmetry

In this and next appendices we present $N = 1$ and $N = 2$ supersymmetries. In our conventions we closely follow [26].

We deal with the real (Majorana) two-component spinors $\psi^{\alpha} = (\psi^{+}, \psi^{-})$. Spinor indices are raised and lowered by the second-rank antisymmetric symbol $C_{\alpha\beta}$, which defines the spinor inner product:

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta}\,, \qquad C_{+-} = i\,, \qquad \psi_{\alpha} = \psi^{\beta} C_{\beta\alpha}\,, \qquad \psi^{\alpha} = C^{\alpha\beta}\psi_{\beta}\,. \tag{A.1}$$

Throughout the paper we use $(\mathbin{+\!\!+}, =)$ as worldsheet indices, and $(+, -)$ as two-dimensional spinor indices. We also use superspace conventions where the pair of spinor coordinates of the two-dimensional superspace are labelled $\theta^{\pm}$, and the covariant derivatives $D_{\pm}$ and supersymmetry generators $Q_{\pm}$ satisfy

$$\begin{aligned} D_{+}^{2} &= i\partial_{\pm\!\!\!+}, \quad D_{-}^{2} = i\partial_{=} \quad \{D_{+}, D_{-}\} = 0 \\ Q_{\pm} &= iD_{\pm} + 2\theta^{\pm}\partial_{\substack{+\\=}} \end{aligned} \tag{A.2}$$

where $\partial_{\substack{+\\=}} = \partial_0 \pm \partial_1$. In terms of the covariant derivatives, a supersymmetry transformation of a superfield Φ is then given by

$$\begin{aligned} \delta\Phi &\equiv i(\varepsilon^{+}Q_{+} + \varepsilon^{-}Q_{-})\Phi & \text{(A.3)} \\ &= -(\varepsilon^{+}D_{+} + \varepsilon^{-}D_{-})\Phi + 2i(\varepsilon^{+}\theta^{+}\partial_{\pm\!\!\!+} + \varepsilon^{-}\theta^{-}\partial_{=})\Phi\,. & \text{(A.4)} \end{aligned}$$

The components of a superfield Φ are defined via projections as follows,

$$\Phi| \equiv X, \quad D_{\pm}\Phi| \equiv \psi_{\pm}, \quad D_{+}D_{-}\Phi| \equiv F_{+-}, \tag{A.5}$$

where a vertical bar denotes "the $\theta = 0$ part of ". Thus, in components, the $(1,1)$ supersymmetry transformations are given by

$$\begin{cases} \delta X^{\mu} = -\epsilon^{+}\psi_{+}^{\mu} - \epsilon^{-}\psi_{-}^{\mu} \\ \delta\psi_{+}^{\mu} = -i\epsilon^{+}\partial_{\pm\!\!\!+}X^{\mu} + \epsilon^{-}F_{+-}^{\mu} \\ \delta\psi_{-}^{\mu} = -i\epsilon^{-}\partial_{=}X^{\mu} - \epsilon^{+}F_{+-}^{\mu} \\ \delta F_{+-}^{\mu} = -i\epsilon^{+}\partial_{\pm\!\!\!+}\psi_{-}^{\mu} + i\epsilon^{-}\partial_{-}\psi_{+}^{\mu}\,. \end{cases} \tag{A.6}$$

The $N = 1$ spinorial mesure in terms of covariant derivatives is

$$\int d^2\theta\ \mathcal{L} = D_{+}D_{-}\mathcal{L}|\,. \tag{A.7}$$

B. (2,2) supersymmetry

For $N = 2$ supersymmetry the situation is considerably more involved. Using $N = 1$ formalism we define complex spinor derivatives

$$D_{\alpha} \equiv \frac{1}{2}\left(D_{\alpha}^{1} + iD_{\alpha}^{2}\right), \qquad \bar{D}_{\alpha} = \frac{1}{2}\left(D_{\alpha}^{1} - iD_{\alpha}^{2}\right) \tag{B.1}$$

with the algebra

$$\{D_{+}, \bar{D}_{+}\} = i\partial_{\pm\!\!\!+}\,, \qquad \{D_{-}, \bar{D}_{-}\} = i\partial_{=} \tag{B.2}$$

$$\{D_{\alpha}, D_{\beta}\} = 0\,, \qquad \{\bar{D}_{\alpha}, \bar{D}_{\beta}\} = 0\,. \tag{B.3}$$

We also complexify the spinor coordinates. Thus the covariant derivatives have the following explicit form

$$D_{\pm} = \partial_{\pm} + \frac{i}{2}\bar{\theta}^{\pm}\partial_{\substack{+\\=}}\,, \qquad \bar{D}_{\pm} = \bar{\partial}_{\pm} + \frac{i}{2}\theta^{\pm}\partial_{\substack{+\\=}}\,. \tag{B.4}$$

In terms of the covariant derivatives, the supersymmetry transformations are

$$Q_{\alpha} = iD_{\alpha} + \theta^{\beta}\partial_{\alpha\beta}\,, \qquad \bar{Q}_{\alpha} = i\bar{D}_{\alpha} + \bar{\theta}^{\beta}\partial_{\alpha\beta}\,. \tag{B.5}$$

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The supersymmetry transformation of a superfield Φ is then defined by

$$\delta\Phi = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon^\alpha \bar Q_\alpha\right)\Phi\,. \tag{B.6}$$

A chiral superfield ($\bar D_\pm \Phi = 0$) has components defined via projections as follows

$$\Phi| \equiv X\,, \qquad D_\pm\Phi| \equiv \psi_\pm\,, \qquad D_+D_-\Phi| \equiv F_{+-}\,, \tag{B.7}$$

In components the $(2,2)$ supersymmetry transformations for the chiral multiplet are given by

$$\begin{cases} \delta X^i = -\epsilon^+\psi^i_+ - \epsilon^-\psi^i_- \\ \delta\psi^i_+ = -i\bar\epsilon^+\partial_{\#}X^i + \epsilon^- F^i_{+-} \\ \delta\psi^i_- = -i\bar\epsilon^-\partial_{=}X^i - \epsilon^+ F^i_{+-} \\ \delta F^i_{+-} = -i\bar\epsilon^+\partial_{\#}\psi^i_- + i\bar\epsilon^-\partial_-\psi^i_+\,. \end{cases} \tag{B.8}$$

The $N=2$ spinorial mesure in terms of covariant derivatives is

$$\int d^2\theta\, d^2\bar\theta\ \mathcal{L} = \left(D_+D_-\bar D_+\bar D_- + \frac{i}{2}D_+\bar D_+\partial_= + \frac{i}{2}D_-\bar D_-\partial_{\#} + \frac{1}{4}\partial_{\#}\partial_=\right)\mathcal{L}| \tag{B.9}$$

where the last three terms are purely boundary and thus usually dropped. The supersymmetry variation of a general action is

$$\begin{aligned} \delta S &= i\int d^2\sigma\, d^2\theta\, d^2\bar\theta(\epsilon^\alpha Q_\alpha + \bar\epsilon^\alpha\bar Q_\alpha)\mathcal{L} \\ &= i\int d^2\sigma\left(\epsilon^\alpha D_\alpha + \bar\epsilon^\alpha \bar D_\alpha\right)\left(D_+D_-\bar D_+\bar D_- + \frac{i}{2}D_+\bar D_+\partial_= + \frac{i}{2}D_-\bar D_-\partial_{\#} + \frac{1}{4}\partial_{\#}\partial_=\right)\mathcal{L}| \end{aligned} \tag{B.10}$$

where again we have kept all boundary terms.

C. Submanifolds

In this appendix we summarise the relevant mathematical details on submanifolds of riemannian manifolds. In our use of terminology we closely follow [12].

We first give the definition of a *distribution* on a manifold (or neighbourhood) $\mathcal{M}$. A distribution π of dimension $(p+1)$ on $\mathcal{M}$ is an assignment to each point $X \in \mathcal{M}$ of a $(p+1)$-dimensional subspace π_X of the tangent space $\mathcal{T}_X(\mathcal{M})$. The assignment can be done in different ways, for instance by means of an appropriate projection operator. π is called *differentiable* if every point X has a neighbourhood U and $(p+1)$ differentiable vector fields, which form a basis of π_Y at every $Y \in U$. π is called *involutive* if for any two vector fields v_i, $v_j \in \pi_X$ their Lie bracket $\{v_i, v_j\} \in \pi_X$ for all $X \in \mathcal{M}$.

A connected submanifold D of $\mathcal{M}$ is called an *integral manifold* of π if $f_*(\mathcal{T}_X(D)) = \pi_X$ for all $X \in D$, where f is the embedding of D into $\mathcal{M}$. If there is no other integral manifold of π which contains D, then D is called a *maximal integral manifold* of π.

Theorem 1 (Frobenius) *Let π be an involutive distribution on a manifold $\mathcal{M}$. Then through every point $X \in \mathcal{M}$, there passes a unique maximal integral manifold $D(X)$ of π. Any other integral manifold through X is an open submanifold of $D(X)$.*

Now we can defined the distribution by means of the projectors. Let $Q^\mu{}_\nu(X)$ be a differentiable distribution[6] which assigns to a point X in the d-dimensional spacetime manifold $\mathcal{M}$ a $(d-p-1)$-dimensional subspace[7] of the tangent space $\mathcal{T}_X(\mathcal{M})$. This subspace consists of all vectors $v^\mu(X) \in \mathcal{T}_X(\mathcal{M})$ such that

$$Q^\mu{}_\nu(X) v^\nu(X) = v^\mu(X)\,. \tag{C.1}$$

The complementary distribution is defined as $\pi^\mu{}_\nu = \delta^\mu{}_\nu - Q^\mu{}_\nu$, and assigns to X a $(p+1)$-dimensional space that consists of vectors $v^\mu(X) \in T_X(\mathcal{M})$ such that

$$\pi^\mu{}_\nu(X) v^\nu(X) = v^\mu(X)\,. \tag{C.2}$$

Now we ask when the vector fields defined by (C.2) form a submanifold. The Lie bracket of two vector fields v and w in π-space is

$$\{v,w\}^\nu = \pi^\nu{}_\sigma \{v,w\}^\sigma + v^\rho w^\sigma \pi^\mu{}_\sigma \pi^\lambda{}_\rho Q^\nu{}_{[\lambda,\mu]}\,. \tag{C.3}$$

If the last term vanishes $\pi^\mu{}_\sigma \pi^\lambda{}_\rho Q^\nu{}_{[\lambda,\mu]} = 0$ (i.e., if $\pi^\mu{}_\nu$ is integrable), then the distribution $\pi^\mu{}_\nu$ is involutive, and due to the Frobenius theorem there is a unique *maximal integral submanifold* corresponding to $\pi^\mu{}_\nu$.

If the manifold $\mathcal{M}$ is riemannian, then various structures may be induced on the submanifold D. For instance, D is automatically riemannian. If one defines the Levi-Civita connection $\nabla_v \equiv v^\mu \nabla_\mu$ on $\mathcal{M}$, and takes two vector fields v and w in the tangent space $\mathcal{T}(D)$ of D, then the covariant derivative $\nabla_v w$ can be decomposed as

$$\nabla_v w = \hat{\nabla}_v w + \mathcal{B}(v,w)\,, \tag{C.4}$$

where $\hat{\nabla}_v w$ is the tangential component (i.e., it is in $\mathcal{T}(D)$) and $\mathcal{B}(v,w)$ is the normal component. One can show that $\hat{\nabla}_v$ can serve as the induced connection on the submanifold D. $\mathcal{B}$ is called the *second fundamental form* of D. Sometimes it is useful to introduce the *associated second fundamental form*, $\mathcal{A}$, which is defined as follows. Taking z to be a normal vector field on D and v a tangent vector field on D we write

$$\nabla_v z = -\mathcal{A}_z v + D_v z \tag{C.5}$$

where $-\mathcal{A}_z v$ and $D_v z$ are, respectively, the tangential and the normal components of $\nabla_v z$. Using the metric g on $\mathcal{M}$ one can prove the following simple identity,

$$g(\mathcal{B}(v,w),z) = g(\mathcal{A}_z v, w)\,. \tag{C.6}$$

Equations (C.4) and (C.5) are called the Gauss formula and the Weingarten formula, respectively. A submanifold D is said to be *totally geodesic* if its second fundamental form vanishes identically, that is, $\mathcal{B} = 0$ or equivalently $\mathcal{A} = 0$.

[6] We need to assume differentiability to be able to do the calculations.

[7] We take $\mathrm{rank}(Q) = d - p - 1$ in order to match the D-brane terminology.

Now we can rewrite the above definitions in terms of the projectors. Take two vector fields v, w in the π-space. Denoting by ∇ the connection on $\mathcal{M}$, we may write

$$v^\mu \nabla_\mu w^\nu = \pi^\nu{}_\rho v^\mu \nabla_\mu w^\rho + Q^\nu{}_\rho v^\mu \nabla_\mu w^\rho \,, \tag{C.7}$$

where we decomposed the derivative into its tangential (to the π-space) and normal parts by using $\delta^\mu_\nu = \pi^\mu{}_\nu + Q^\mu{}_\nu$. The tangential component is the induced connection, and the normal component is the second fundamental form (C.4). The latter may be rewritten, using that $\pi^\mu{}_\nu v^\nu = v^\mu$ and $\pi^\mu{}_\nu w^\nu = w^\mu$, as

$$v^\delta w^\sigma \, \mathcal{B}^\lambda{}_{\delta\sigma} \equiv -v^\delta w^\sigma \; \pi^\mu{}_\delta \pi^\nu{}_\sigma \nabla_\mu Q^\lambda{}_\nu \,. \tag{C.8}$$

Note that $\mathcal{B}^\lambda{}_{\delta\sigma}$ is symmetric in the indices δ and σ, as a second fundamental form must be, because $\pi^\mu{}_\nu$ is integrable.

Performing the same decomposition for the derivative of a vector field u in the Q-space, we have

$$v^\mu \nabla_\mu u^\nu = \pi^\nu{}_\rho v^\mu \nabla_\mu u^\rho + Q^\nu{}_\rho v^\mu \nabla_\mu u^\rho \,, \tag{C.9}$$

where $Q^\mu{}_\nu u^\nu = u^\mu$ and v is in the π-space, $\pi^\mu{}_\nu v^\nu = v^\mu$. The associated second fundamental form is then defined as the tangential part, which we can rewrite as

$$v^\delta u^\sigma \, \mathcal{A}^\lambda{}_{\sigma\delta} \equiv -v^\delta u^\sigma \; \pi^\mu{}_\delta \pi^\lambda{}_\nu \nabla_\mu Q^\nu{}_\sigma \,. \tag{C.10}$$

Thus the manifold is totally geodesic if and only if one of the equivalent properties holds

$$\pi^\mu{}_\delta \pi^\lambda{}_\nu \nabla_\mu Q^\nu{}_\sigma = 0 \quad \text{or} \quad \pi^\mu{}_\delta \pi^\nu{}_\sigma \nabla_\mu Q^\lambda{}_\nu = 0 \tag{C.11}$$

If on a manifold $\mathcal{M}$ there is $(1,1)$ non-null tensor field, L then one can consider the invariant submanifolds under the action of L, i.e.

$$L\mathcal{T}_X(D) \subset \mathcal{T}_X(D) \,, \qquad L\mathcal{N}_X(D) \subset \mathcal{N}_X(D) \tag{C.12}$$

and anti-invariant submanifolds under the action L, i.e.

$$L\mathcal{T}_X(D) \subset \mathcal{N}_X(D) \,, \qquad L\mathcal{N}_X(D) \subset \mathcal{T}_X(D) \,. \tag{C.13}$$

D. Symplectic geometry

A manifold is called *symplectic* if there exists a nondegenerate closed two-form ω. This two-form is a symplectic form. In order to classify the submanifold of a symplectic manifold one has to classify the subspaces of the symplectic linear space.

Let W be a k-dimensional subspace of the $2n$-dimensional symplectic space (V, ω). Then $k = dim(W)$ and $2l = rank(\omega|_W)$ remain unchanged under any symplectic morphism from $\mathrm{Sp}(V)$. Therefore these two integers, k and $2l$, classify the subspaces of V and they are the only two independent symplectic invariants for subspaces. The most important cases of subspaces of a symplectic space (V, ω) are the following.

- A subspace $W \subset V$ with $\omega|_W$ non-degenerate is called a *symplectic subspace.*

- A subspace $W \subset V$ with $\omega|_W = 0$ is called an *isotropic subspace.*
- A subspace $W \subset V$ with $W^\perp$ isotropic is called *coisotropic.*
- A subspace $W \subset V$ which is both isotropic and coisotropic is called a *lagrangian subspace.*

In the above definitions $W^\perp$ is the ω-orthogonal space

$$W^\perp \equiv \{v \in V; \omega(v,w) = 0, \forall w \in W\}$$

Alternatively, one can give the follwing definitions for a subspace $W \subset V$ with $k = dim(W)$

$$\begin{array}{rclcl} W \quad \text{isotropic} & \Leftrightarrow & W \subset W^\perp & \rightarrow & k \leq n \\ W \quad \text{coisotropic} & \Leftrightarrow & W \supset W^\perp & \rightarrow & k \geq n \\ W \quad \text{Lagrange} & \Leftrightarrow & W = W^\perp & \rightarrow & k = n\,. \end{array} \tag{D.1}$$

When this classification is applied to the tangent space $\mathcal{T}(D)$ of a submanifold D of a symplectic manifold $\mathcal{M}$ we have the classification of submanifolds of a symplectic manifold.

For a general introduction to symplectic geometry the reader can consult, for example, the following book [27].

E. F-structures

A structure on an d-dimensional manifold $\mathcal{M}$ given by a non-null tensor field f satisfying

$$f^3 + f = 0 \tag{E.1}$$

is called an f-structure (due to Yano). Then the rank of f is constant. If $d = \text{rank}(f)$, then an f-structure gives an almost complex structure (i.e., $f^2 = -I$) of the manifold $\mathcal{M}$. If $\mathcal{M}$ is orientable and $d - 1 = \text{rank}(f)$, then an f-structure gives an almost contact structure of the manifold $\mathcal{M}$. An f-structure is called integrable if the Nijenhuis tensor of f is zero.

Theorem 2. *A necesary and sufficient condition for a d-dimensional manifold $\mathcal{M}$ to admit an f-structure f is that the rank of f is even,* $\text{rank}(f) = 2m$*, and that the group of the tangent bundle of $\mathcal{M}$ be reduced to the group* $\mathrm{U}(m) \times O(d-2m)$*.*

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